

# DEGENERATIONS OF CALABI-YAU THREEFOLDS AND BCOV INVARIANTS

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ABSTRACT. In [1], [2], by expressing the physical quantity  $F_1$  in two distinct ways, Bershadsky-Cecotti-Ooguri-Vafa discovered a remarkable equivalence between Ray-Singer analytic torsion and elliptic instanton numbers for Calabi-Yau threefolds. After their discovery, in [7], a holomorphic torsion invariant for Calabi-Yau threefolds corresponding to  $F_1$ , called BCOV invariant, was constructed. In this article, we study the asymptotic behavior of BCOV invariants for algebraic one-parameter degenerations of Calabi-Yau threefolds. We prove the rationality of the coefficient of logarithmic divergence and give its geometric expression by using a semi-stable reduction of the given family.

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## INTRODUCTION

In [1], [2], by expressing the physical quantity  $F_1$  in two distinct ways, Bershadsky-Cecotti-Ooguri-Vafa discovered a remarkable equivalence between Ray-Singer analytic torsion and elliptic instanton numbers for Calabi-Yau threefolds. After their discovery, in [7], a holomorphic torsion invariant for Calabi-Yau threefolds corresponding to  $F_1$ , called *BCOV invariant*, was constructed. Because of its invariance property, BCOV invariant gives rise to a function  $\tau_{\text{BCOV}}$  on the moduli space of Calabi-Yau threefolds. In physics literatures,  $-\log \tau_{\text{BCOV}}$  is denoted by  $F_1$ . The prediction of Bershadsky-Cecotti-Ooguri-Vafa concerning the equivalence of holomorphic torsion and elliptic instanton numbers for Calabi-Yau threefolds can be stated as follows:  $\tau_{\text{BCOV}}$  admits an explicit infinite product expression of Borchers type near the large complex structure limit point of the compactified moduli space of Calabi-Yau threefolds, and the exponents of the infinite product are given by explicit linear combinations of rational and elliptic instanton numbers of the mirror Calabi-Yau threefold corresponding to the large complex structure limit point.

As was done in [1], [2], a possible first step towards the conjecture of Bershadsky-Cecotti-Ooguri-Vafa is to determine the section of certain holomorphic line bundle

on the moduli space corresponding to  $\tau_{\text{BCOV}}$ . Thanks to the curvature theorem of Bismut-Gillet-Soulé [4], the complex Hessian  $dd^c \log \tau_{\text{BCOV}}$  is expressed as an explicit linear combination of the Weil-Petersson form and its Ricci-form on the moduli space [1], [2], [6], [7]. However, since the moduli space of Calabi-Yau threefolds are non-compact in general, the complex Hessian  $dd^c \log \tau_{\text{BCOV}}$  does not determine uniquely its potential and hence the corresponding holomorphic section. To determine its potential up to a constant,  $dd^c \log \tau_{\text{BCOV}}$  must be determined as a current on some compactified moduli space. In this way, we are led to the following two problems: one is to understand the behaviors of Weil-Petersson and its Ricci forms as well as their potentials near the boundary locus of the compactified moduli space; the other is to understand the behavior of  $\tau_{\text{BCOV}}$  near the boundary locus of the compactified moduli space. We refer to [10], [11], [7], [12] for the first problem. In this article, we focus on the second problem.

In this direction, in [7], the following results were obtained as an application of the theory of Quillen metrics [5], [3], [18]:  $\log \tau_{\text{BCOV}}$  always has logarithmic singularity for arbitrary algebraic one-parameter degenerations of Calabi-Yau threefolds and the logarithmic singularity of  $\log \tau_{\text{BCOV}}$  is determined for smoothings of Calabi-Yau varieties with at most one ordinary double point under an additional assumption of the dimension of moduli space. These results, together with the formula for  $dd^c \log \tau_{\text{BCOV}}$  and the known boundary behaviors of Weil-Petersson and its Ricci forms, are sufficient to determine  $\tau_{\text{BCOV}}$  for quintic mirror threefolds [7]. However, the results in [7] concerning the singularity of  $\tau_{\text{BCOV}}$  are not sufficient to determine an explicit formula for  $\tau_{\text{BCOV}}$  for wider classes of Calabi-Yau threefolds, e.g. Calabi-Yau threefolds of Borcea-Voisin. For this reason, it is strongly desired to improve the above results in [7]. The purpose of the present article is to give such improvements. Let us explain our main results.

Let  $f: \mathcal{X} \rightarrow C$  be a surjective morphism from an irreducible projective fourfold  $\mathcal{X}$  to a compact Riemann surface  $C$ . Assume that there exists a finite subset  $\Delta_f \subset C$  such that  $f|_{C \setminus \Delta_f}: \mathcal{X}|_{C \setminus \Delta_f} \rightarrow C \setminus \Delta_f$  is a smooth morphism and such that  $X_t = f^{-1}(t)$  is a Calabi-Yau threefold for all  $t \in C \setminus \Delta_f$ .

**Theorem 0.1.** *For every  $0 \in \Delta_f$ , there exists  $\alpha = \alpha_0 \in \mathbf{Q}$  such that*

$$\log \tau_{\text{BCOV}}(X_t) = \alpha \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0),$$

where  $t$  is a local parameter of  $C$  centered at  $0 \in \Delta_f$ .

We remark that in the corresponding theorem in [7], the rationality of  $\alpha$  was missing. After Theorem 0.1, a natural question is how the coefficient  $\alpha$  is determined by the family  $f: \mathcal{X} \rightarrow C$ . For this, following [7], we consider its semi-stable reduction [13]. Let  $g: (\mathcal{Y}, Y_0) \rightarrow (B, 0)$  be a semi-stable reduction of  $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$ . By definition,  $\mathcal{Y}$  is a smooth projective fourfold,  $(B, 0)$  is a pointed compact Riemann surface and there is a surjective morphism of pointed compact Riemann surfaces  $\phi: (B, 0) \rightarrow (C, 0)$  such that  $Y_0 = g^{-1}(0)$  is a *reduced* normal crossing divisor of  $\mathcal{Y}$  and such that  $\mathcal{Y} \setminus Y_0 \cong (\mathcal{X} \setminus X_0) \times_{C \setminus \{0\}} (B \setminus \{0\})$ .

By choosing a small neighborhood  $V$  of  $0$  in  $B$ ,  $g^{-1}(V)$  carries a canonical form whose zero divisor is contained in  $Y_0$ . The zero divisor of any canonical form on  $g^{-1}(V)$  with this property is independent of the choice of such canonical form and is denoted by  $\mathfrak{K}_{(\mathcal{Y}, Y_0)}$ . We call  $\mathfrak{K}_{(\mathcal{Y}, Y_0)}$  the normalized canonical divisor.

Let  $\Omega_{\mathcal{Y}/B}^1$  be the sheaf of relative Kähler differentials on  $\mathcal{Y}$  and let  $\Omega_{\mathcal{Y}/B}^1(\log)$  be its logarithmic version. Set  $\mathcal{Q} = \Omega_{\mathcal{Y}/B}^1(\log)/\Omega_{\mathcal{Y}/B}^1$ . Every direct image  $R^q g_* \mathcal{Q}|_V$  is

a finitely generated torsion sheaf on  $V$  supported at  $0 \in B$ . We set  $\chi(Rg_*\mathcal{Q}|_V) = \sum_{q \geq 0} (-1)^q \dim_{\mathbf{C}}(R^q g_* \mathcal{Q})_0 \in \mathbf{Z}$ .

Let  $\Sigma_g$  be the critical locus of  $g$ . Let  $\mathbf{P}(T\mathcal{Y})^\vee$  be the projective bundle over  $\mathcal{Y}$  whose fiber  $\mathbf{P}(T\mathcal{Y})_y^\vee$  is the projective space of hyperplanes of  $T_y \mathcal{Y}$ . Then the Gauss map  $\mu: \mathcal{Y} \setminus \Sigma_g \ni y \rightarrow [T_y Y_{g(y)}] \in \mathbf{P}(T\mathcal{Y})^\vee$  extends to a meromorphic map from  $\mathcal{Y}$  to  $\mathbf{P}(T\mathcal{Y})^\vee$ . Namely, there exists a blowing-up  $\sigma: \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  inducing an isomorphism  $\tilde{\mathcal{Y}} \setminus \sigma^{-1}(\Sigma_g) \cong \mathcal{Y} \setminus \Sigma_g$  such that the composite  $\tilde{\mu} = \mu \circ \sigma$  extends to a holomorphic map from  $\tilde{\mathcal{Y}}$  to  $\mathbf{P}(T\mathcal{Y})^\vee$ . Set  $\tilde{g} = g \circ \sigma$ . We have a new family of Calabi-Yau threefolds  $\tilde{g}: \tilde{\mathcal{Y}} \rightarrow B$ , whose critical locus  $\Sigma_{\tilde{g}}|_V$  defines a divisor of  $\tilde{\mathcal{Y}}$ .

Let  $U$  be the universal hyperplane bundle over  $\mathbf{P}(T\mathcal{Y})^\vee$  and let  $H$  be the universal quotient line bundle over  $\mathbf{P}(T\mathcal{Y})^\vee$ . Following [7], set

$$a_p(g, \Sigma_g) = \sum_{j=0}^p (-1)^{p-j} \int_{\text{Exc}(\sigma)} \tilde{\mu}^* \left\{ \text{Td}(U) \frac{\text{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} \sigma^* \text{ch}(\Omega_{\mathcal{Y}}^j),$$

where  $\text{Exc}(\sigma)$  is the exceptional divisor of  $\sigma: \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ ,  $\text{Td}(\cdot)$  is the Todd genus, and  $\Omega_{\mathcal{Y}}^j$  is the holomorphic vector bundle of holomorphic  $j$ -forms on  $\mathcal{Y}$ . Define

$$\rho(g, \Sigma_g) = -3a_0(g, \Sigma_g) + 2a_1(g, \Sigma_g) - \chi(Rg_*\mathcal{Q}|_V) + \frac{1}{12} \int_{\Sigma_{\tilde{g}}|_V} \tilde{\mu}^* c_3(U) \in \mathbf{Q},$$

$$\kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) = \int_{\sigma^* \mathfrak{K}_{(\mathcal{Y}, Y_0)}} \tilde{\mu}^* c_3(U) \in \mathbf{Z}.$$

**Theorem 0.2.** *The rational number  $\alpha$  in Theorem 0.1 is given by*

$$\alpha = \frac{1}{\deg\{\phi: (B, 0) \rightarrow (C, 0)\}} \left\{ \rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) \right\}.$$

Since every algebraic one-parameter degeneration of Calabi-Yau threefolds admits a semi-stable reduction [13], in principle, one can compute the singularity of  $\tau_{\text{BCOV}}$  for those degenerations by Theorems 0.1 and 0.2, once one knows their semi-stable reductions. In this sense, the problem of understanding the singularity of  $\tau_{\text{BCOV}}$  is reduced to the algebro-geometric problem of classifying possible semi-stable degenerations of Calabi-Yau threefolds.

As an application of Theorems 0.1 and 0.2, we shall prove certain locality of the singularity of  $\tau_{\text{BCOV}}$ . Namely, under some additional assumptions about the family  $f: \mathcal{X} \rightarrow C$  (cf. Section 4 for the required conditions), the coefficient  $\alpha$  in Theorem 0.1 depends only on the function germ of  $f$  around the critical locus  $\Sigma_f$ . (See Theorem 4.1 for the precise statement.) In some cases, this locality is quite powerful, because we have only to compute one particular example to determine the singularity of  $\tau_{\text{BCOV}}$ . This locality result plays a crucial role to determine the BCOV invariant for Borcea-Voisin threefolds [21].

The strategy to the proof of Theorems 0.1 and 0.2 is quite parallel to that of [7, Th. 9.1]. In [7], it was proved that the  $L^2$ -metric on the line bundle  $\det R^q g_* \Omega_{\mathcal{Y}/B}^p(\log)$  has at most an algebraic singularity at the discriminant locus when  $p + q = 3$ . In this article, we shall improve this estimate. Namely, under the assumption of semi-stability, the  $L^2$ -metric on  $\det R^q g_* \Omega_{\mathcal{Y}/B}^p(\log)$  has at most a logarithmic singularity, which enables us to determine various inexplicit constants in [7, §9] and hence  $\alpha$  in Theorem 0.1.

This article is organized as follows. In Section 1, we recall the construction of BCOV invariants. In Section 2, we study the asymptotic behavior of the  $L^2$ -metric on  $\det R^q g_* \Omega_{Y/B}^p(\log)$  and prove the key fact that it has at most a mild singularity when  $p + q = 3$ . In Section 3, we prove Theorems 0.1 and 0.2. In Section 4, we prove the locality of the singularity of  $\tau_{\text{BCOV}}$ . In Section 5, we determine the singularity of  $\tau_{\text{BCOV}}$  for general one-parameter smoothings of Calabi-Yau varieties with at most ordinary double points.

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## 1. BCOV INVARIANTS

**1.1. Analytic torsion and Quillen metrics.** Let  $(M, g)$  be a compact Kähler manifold of dimension  $d$  with Kähler form  $\omega$ . Let  $\square_{p,q} = (\bar{\partial} + \bar{\partial}^*)^2$  be the Hodge-Kodaira Laplacian acting on  $C^\infty$   $(p, q)$ -forms on  $M$  or equivalently  $(0, q)$ -forms on  $M$  with values in  $\Omega_M^p$ , where  $\Omega_M^1$  is the holomorphic cotangent bundle of  $M$  and  $\Omega_M^p := \wedge^p \Omega_M^1$ . Let  $\sigma(\square_{p,q}) \subset \mathbf{R}_{\geq 0}$  be the set of eigenvalues of  $\square_{p,q}$ . The spectral zeta function of  $\square_{p,q}$  is defined as

$$\zeta_{p,q}(s) := \sum_{\lambda \in \sigma(\square_{p,q}) \setminus \{0\}} \lambda^{-s} \dim E(\lambda, \square_{p,q}),$$

where  $E(\lambda, \square_{p,q})$  is the eigenspace of  $\square_{p,q}$  corresponding to the eigenvalue  $\lambda$ . Then  $\zeta_{p,q}(s)$  converges on the half-plane  $\{s \in \mathbf{C}; \Re s > \dim M\}$ , extends to a meromorphic function on  $\mathbf{C}$ , and is holomorphic at  $s = 0$ . By Ray-Singer [15], the *analytic torsion* of  $(M, \Omega_M^p)$  is the real number defined as

$$\tau(M, \Omega_M^p) := \exp\left\{-\sum_{q \geq 0} (-1)^q q \zeta'_{p,q}(0)\right\}.$$

Obviously,  $\tau(M, \Omega_M^p)$  depends not only on the complex structure of  $M$  but also on the metric  $g$ . When we emphasize the dependence of analytic torsion on the metric, we write  $\tau(M, \Omega_M^p, g)$ .

In [2], Bershadsky-Cecotti-Ooguri-Vafa introduced the following combination of analytic torsions.

**Definition 1.1.** The *BCOV torsion* of  $(M, g)$  is the real number defined as

$$T_{\text{BCOV}}(M, g) := \prod_{q \geq 0} \tau(M, \Omega_M^p)^{(-1)^p} = \exp\left\{-\sum_{p,q \geq 0} (-1)^{p+q} pq \zeta'_{p,q}(0)\right\}.$$

If  $\gamma$  is the Kähler form of  $g$ , then we often write  $T_{\text{BCOV}}(M, \gamma)$  for  $T_{\text{BCOV}}(M, g)$ . In general,  $T_{\text{BCOV}}(M, g)$  does depend on the choice of Kähler metric  $g$  and hence is not a holomorphic invariant of  $M$ . When  $M$  is a Calabi-Yau threefold, it is possible to construct a holomorphic invariant of  $M$  from  $T_{\text{BCOV}}(M, g)$  by multiplying a correction factor. Following [7], let us recall the construction of this invariant.

**1.2. Calabi-Yau threefolds and BCOV invariants.** A compact connected Kähler manifold  $X$  is *Calabi-Yau* if  $h^{0,q}(X) = 0$  for  $0 < q < \dim X$  and  $K_X \cong \mathcal{O}_X$ , where  $K_X$  is the canonical line bundle of  $X$ . Our particular interest is the case where  $X$  is a threefold. Let  $X$  be a Calabi-Yau threefold. Let  $g = \sum_{i,j} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$  be a

Kähler metric on  $X$  and let  $\gamma = \gamma_g := \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$  be the corresponding Kähler form. Following the convention in Arakelov geometry, we define

$$\text{Vol}(X, \gamma) := \frac{1}{(2\pi)^3} \int_X \frac{\gamma^3}{3!}.$$

The covolume of  $H^2(X, \mathbf{Z})_{\text{free}} := H^2(X, \mathbf{Z})/\text{Torsion}$  with respect to  $[\gamma]$  is defined as

$$\text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma]) := \det(\langle \mathbf{e}_i, \mathbf{e}_j \rangle_{L^2, [\gamma]})_{1 \leq i, j \leq b_2(X)}.$$

Here  $\{\mathbf{e}_1, \dots, \mathbf{e}_{b_2(X)}\}$  is a basis of  $H^2(X, \mathbf{Z})_{\text{free}} = \text{Im}\{H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathbf{R})\}$  over  $\mathbf{Z}$  and  $\langle \cdot, \cdot \rangle_{L^2, [\gamma]}$  is the inner product on  $H^2(X, \mathbf{R})$  induced by integration of harmonic forms. Namely, if  $\mathcal{H}\mathbf{e}_i$  denotes the harmonic representative of  $\mathbf{e}_i \in H^2(X, \mathbf{R})$  with respect to  $\gamma$  and if  $*$  denotes the Hodge star operator, then

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle_{L^2, [\gamma]} := \frac{1}{(2\pi)^3} \int_X \mathcal{H}\mathbf{e}_i \wedge *(\mathcal{H}\mathbf{e}_j).$$

The covolume  $\text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])$  is the volume of real torus  $H^2(X, \mathbf{R})/H^2(X, \mathbf{Z})_{\text{free}}$  with respect to the  $L^2$ -metric  $\langle \cdot, \cdot \rangle_{L^2, [\gamma]}$  on  $H^2(X, \mathbf{R})$ .

As the correction term to the BCOV torsion  $T_{\text{BCOV}}(X, \gamma)$ , we introduce a Bott-Chern term.

**Definition 1.2.** For a Calabi-Yau threefold  $X$  equipped with a Kähler form, define

$$A(X, \gamma) := \exp \left[ -\frac{1}{12} \int_X \log \left( \sqrt{-1} \frac{\eta \wedge \bar{\eta}}{\gamma^3/3!} \frac{\text{Vol}(X, \gamma)}{\|\eta\|_{L^2}^2} \right) c_3(X, \gamma) \right],$$

where  $c_3(X, \gamma)$  is the top Chern form of  $(X, \gamma)$ ,  $\eta \in H^0(X, K_X) \setminus \{0\}$  is a nowhere vanishing canonical form on  $X$  and  $\|\eta\|_{L^2}$  is its  $L^2$ -norm, i.e.,

$$\|\eta\|_{L^2} := \frac{1}{(2\pi)^3} \int_X \sqrt{-1} \eta \wedge \bar{\eta}.$$

Obviously,  $A(X, \gamma)$  is independent of the choice of  $\eta \in H^0(X, K_X) \setminus \{0\}$ . We remark that our definition of  $A(X, \gamma)$  differs from the one in [7, Def. 4.1] by the factor  $\text{Vol}(X, \gamma)^{\chi(X)/12}$ , where  $\chi(X)$  denotes the topological Euler number of  $X$ . Notice that  $A(X, \gamma) = 1$  if  $\gamma$  is Ricci-flat.

**Definition 1.3.** The *BCOV invariant* of  $X$  is the real number defined as

$$\tau_{\text{BCOV}}(X) := \text{Vol}(X, \gamma)^{-3 + \frac{\chi(X)}{12}} \text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])^{-1} T_{\text{BCOV}}(X, \gamma) A(X, \gamma).$$

As an application of the curvature formula for Quillen metrics [4, Th. 0.1], we get the invariance property of  $\tau_{\text{BCOV}}(X)$  in [7].

**Theorem 1.4.** *For a Calabi-Yau threefold  $X$ ,  $\tau_{\text{BCOV}}(X)$  is independent of the choice of a Kähler form on  $X$ .*

*Proof.* See [7, Th. 4.16]. □

After Theorem 1.4, we regard  $\tau_{\text{BCOV}}$  as a function on the moduli space of Calabi-Yau threefolds. In this article, we study the behavior of  $\tau_{\text{BCOV}}$  for algebraic one-parameter families of Calabi-Yau threefolds and improve some results in [7, §9].

## 2. ASYMPTOTIC BEHAVIOR OF THE $L^2$ -METRIC ON HODGE BUNDLE

Let  $\Delta := \{z \in \mathbf{C}; |z| < 1\}$  be the unit disc and let  $\Delta^* := \Delta \setminus \{0\}$  be the unit punctured disc. Let  $\mathfrak{H} := \{z \in \mathbf{C}; \Im z > 0\}$  be the complex upper half-plane. We regard  $\mathfrak{H}$  as the universal covering of  $\Delta^*$  by the map  $\varpi: \mathfrak{H} \ni z \rightarrow \exp(2\pi iz) \in \Delta^*$ .

Let  $f: \mathcal{Z} \rightarrow \Delta$  be a proper surjective holomorphic map from a smooth complex manifold of dimension  $n+1$ . We set  $Z_t := f^{-1}(t)$  for  $t \in \Delta$ . If  $Z_t$  is smooth for all  $t \in \Delta^*$  and if  $Z_0$  is a reduced normal crossing divisor of  $\mathcal{Z}$ , then the family  $f: \mathcal{Z} \rightarrow \Delta$  is called a *semi-stable degeneration* of relative dimension  $n$ .

Let  $f: \mathcal{Z} \rightarrow \Delta$  be a semi-stable degeneration of relative dimension  $n$ . Set  $f^o := f|_{\Delta^*}$  and  $\mathcal{Z}^o := \mathcal{Z} \setminus Z_0$ . Let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{Z}$ . We consider the cohomology of middle degree and set  $\ell := \dim H^n(Z_t, \mathbf{C})$  for  $t \neq 0$ . Assume that

$H^n(Z_t, \mathbf{C})$  consists of primitive cohomology classes with respect to  $c_1(\mathcal{L})|_{Z_t}$ .

By the primitivity, each component  $H^{p,q}(Z_t)$ ,  $p+q = n$ , carries the  $L^2$ -inner product

$$(2.1) \quad (u, v)_{L^2, t} := (\sqrt{-1})^{p-q} (-1)^{\frac{n(n-1)}{2}} \int_{Z_t} u \wedge \bar{v}.$$

In particular,  $H^n(Z_t, \mathbf{C})$  is endowed with the  $L^2$ -Hermitian structure, which is independent of the choice of polarization.

**2.1.  $L^2$ -length of flat section.** The holomorphic vector bundle  $R^n f_*^o \mathbf{C} \otimes \mathcal{O}_{\Delta^*}$  is endowed with the Gauss-Manin connection. Fix a reference point  $t_0 \in \Delta$  and set  $V := H^n(Z_{t_0}, \mathbf{C})$ . Fix a basis  $\{v_1, \dots, v_\ell\}$  of  $V$ , which is unitary with respect to the  $L^2$ -inner product at  $t = t_0$ . Since  $\mathfrak{H}$  is simply connected,  $\varpi^*(R^n f_*^o \mathbf{C})$  is a trivial local system of rank  $\ell$  over  $\mathfrak{H}$ . By fixing a point  $z_0 \in \mathfrak{H}$  with  $t_0 = \varpi(z_0)$ , each  $v_i \in \varpi^*(R^n f_*^o \mathbf{C})|_{z_0}$  extends uniquely to a flat section  $\mathbf{v}_i$  of  $\varpi^*(R^n f_*^o \mathbf{C}) = V \times \mathfrak{H}$  with respect to the Gauss-Manin connection such that  $\mathbf{v}_i(z_0) = v_i$ . We regard  $\mathbf{v}_i(z)$  as a  $V$ -valued holomorphic function on  $\mathfrak{H}$ . Namely,  $\mathbf{v}_i(z) \in \mathcal{O}(\mathfrak{H}) \otimes_{\mathbf{C}} V$ .

On  $\varpi^*(R^n f_*^o \mathbf{C}) = V \times \mathfrak{H}$ , the generator of  $\pi_1(\Delta^*) = \mathbf{Z}$  acts as the Picard-Lefschetz transformation: There exists  $T \in \text{Aut}(V)$  such that for all  $z \in \mathfrak{H}$ ,

$$\mathbf{v}_i(z+1) = T\mathbf{v}_i(z) \quad (i = 1, \dots, \ell).$$

Since  $\mathbf{v}_1(z), \dots, \mathbf{v}_\ell(z)$  are flat with respect to the Gauss-Manin connection, there exist constants  $t_{ij}$ ,  $1 \leq i, j \leq \ell$ , such that  $T\mathbf{v}_i(z) = \sum_j t_{ij} \mathbf{v}_j(z)$ . Since  $f: \mathcal{Z} \rightarrow \Delta$  is a semi-stable degeneration,  $T = (t_{ij})$  is unipotent.

Let  $(\cdot, \cdot)_{L^2, z}$  be the  $L^2$ -inner product on  $H^n(Z_{\varpi(z)}, \mathbf{C})$ . Under the identification of  $H^n(Z_{\varpi(z)}, \mathbf{C})$  with  $V = H^n(Z_{\varpi(z_0)}, \mathbf{C})$  via the Gauss-Manin connection,  $(\cdot, \cdot)_{L^2, z}$  is regarded as a family of Hermitian structures on  $V$  such that

$$(2.2) \quad (\mathbf{v}(z+1), \mathbf{v}'(z+1))_{L^2, z+1} = (T\mathbf{v}(z), T\mathbf{v}'(z))_{L^2, z}$$

for any flat sections  $\mathbf{v}(z), \mathbf{v}'(z) \in \mathcal{O}(\mathfrak{H}) \otimes_{\mathbf{C}} V$  with respect to the Gauss-Manin connection.

Let  $a < b$  be real numbers with  $0 < b - a < 1$ . By [8, p.46 Prop. 25], there exist positive constants  $C_1 = C_1(a, b)$ ,  $C_2 = C_2(a, b)$ ,  $k = k(a, b) > 0$  such that for all  $z \in \mathfrak{H}$  with  $a < \Re z < b$ ,  $\Im z \gg 0$  and for all  $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbf{C}^\ell$ , one has

$$(2.3) \quad C_1 (\Im z)^{-k} \|\mathbf{c}\|^2 \leq \left\| \sum_{i=1}^{\ell} c_i \mathbf{v}_i(z) \right\|_{L^2, z}^2 \leq C_2 (\Im z)^k \|\mathbf{c}\|^2,$$

where  $\|\mathbf{c}\|^2 = \sum_{i=1}^{\ell} |c_i|^2 = \left\| \sum_{i=1}^{\ell} c_i \mathbf{v}_i(z) \right\|_{L^2, z_0}^2$ .

*Remark 2.1.* By the  $\mathrm{SL}_2$ -orbit theorem of Schmid [16, Th. 6.6 and its proof] (in particular, the last equality of [16, p. 253]), there is an integer  $\nu(\mathbf{c}) \in \mathbf{Z}_{\geq 0}$  depending on  $\mathbf{c}$  such that the ratio  $\|\sum_{i=1}^{\ell} c_i \mathbf{v}_i(z)\|_{L^2, z}^2 / (\Im z)^{\nu(\mathbf{c})}$  is bounded from below and above by positive constants when  $a < \Re z < b$ ,  $\Im z \gg 0$ . Hence we indeed have the following better estimate

$$(2.4) \quad C_1 \|\mathbf{c}\|^2 \leq \left\| \sum_{i=1}^{\ell} c_i \mathbf{v}_i(z) \right\|_{L^2, z}^2 \leq C_2 (\Im z)^k \|\mathbf{c}\|^2,$$

where  $k \in \mathbf{Z}_{\geq 0}$ . For our later purpose, the weaker estimate (2.3), whose proof is much easier than that of the  $\mathrm{SL}_2$ -orbit theorem, is sufficient.

Let  $H(z)$  be the positive-definite  $\ell \times \ell$ -Hermitian matrix defined as

$$H(z) := \begin{pmatrix} (\mathbf{v}_1(z), \mathbf{v}_1(z))_{L^2, z} & \cdots & (\mathbf{v}_1(z), \mathbf{v}_{\ell}(z))_{L^2, z} \\ \vdots & \ddots & \vdots \\ (\mathbf{v}_{\ell}(z), \mathbf{v}_1(z))_{L^2, z} & \cdots & (\mathbf{v}_{\ell}(z), \mathbf{v}_{\ell}(z))_{L^2, z} \end{pmatrix}.$$

Let  $\lambda_{\min}(z)$  (resp.  $\lambda_{\max}(z)$ ) be the smallest (resp. largest) eigenvalue of the positive-definite Hermitian matrix  $H(z)$ . By (2.3), we get

$$(2.5) \quad C_1 (\Im z)^{-k} \leq \lambda_{\min}(z) \leq \lambda_{\max}(z) \leq C_2 (\Im z)^k.$$

For a multi-index  $I = \{i_1 < i_2 < \cdots < i_r\}$  with  $|I| := r \leq \ell$ , we define

$$\mathbf{v}_I(z) := \mathbf{v}_{i_1}(z) \wedge \mathbf{v}_{i_2}(z) \wedge \cdots \wedge \mathbf{v}_{i_r}(z) \in \Lambda^r V.$$

For  $1 \leq r \leq \ell$ , let  $\Lambda^r H(z)$  be the positive-definite  $\binom{\ell}{r} \times \binom{\ell}{r}$  Hermitian matrix defined as

$$\Lambda^r H(z) := ((\mathbf{v}_I(z), \mathbf{v}_J(z))_{L^2, z})_{|I|=|J|=r}.$$

Here  $\Lambda^r V$  is equipped with the Hermitian structure induced by  $(\cdot, \cdot)_{L^2, z}$ , which is again denoted by the same symbol. By (2.5), we have the following inequality of positive-definite Hermitian endomorphisms on  $\Lambda^r V$

$$(2.6) \quad C_1^r (\Im z)^{-kr} I_{\Lambda^r V} \leq \lambda_{\min}(z)^r I_{\Lambda^r V} \leq \Lambda^r H(z) \leq \lambda_{\max}(z)^r I_{\Lambda^r V} \leq C_2^r (\Im z)^{kr} I_{\Lambda^r V}.$$

By (2.6), for all  $z \in \mathfrak{H}$  with  $a < \Re z < b$  and  $\xi = (\xi_I)_{|I|=r} \in \mathbf{C}^{\binom{\ell}{r}}$ , we get

$$(2.7) \quad C_1^r (\Im z)^{-kr} \|\xi\|^2 \leq \left\| \sum_{|I|=r} \xi_I \mathbf{v}_I(z) \right\|_{L^2, z}^2 \leq C_2^r (\Im z)^{kr} \|\xi\|^2.$$

**2.2.  $L^2$ -length of the canonical section associated to flat section.** From the flat sections  $\mathbf{v}_1(z), \dots, \mathbf{v}_{\ell}(z)$ , one can construct nowhere vanishing  $\pi_1(\Delta^*)$ -invariant holomorphic sections  $\mathbf{s}_1, \dots, \mathbf{s}_{\ell}$  as follows. Let  $N \in \mathrm{End}(V)$  be the logarithm of the Picard-Lefschetz transformation  $T$ . Since  $T$  is unipotent by our assumption, we have  $N = \sum_{l \geq 1} (-1)^{l+1} (T - 1_V)^l / l$ . Since the entries of  $T \in \mathrm{Aut}(V)$  with respect to the basis  $\{\mathbf{v}_1(z), \dots, \mathbf{v}_{\ell}(z)\}$  are constant, so are the entries of  $N \in \mathrm{End}(V)$ . We define

$$(2.8) \quad \mathbf{s}_i(\exp(2\pi\sqrt{-1}z)) := e^{-zN} \mathbf{v}_i(z) = \sum_{k \geq 0} \frac{(-1)^k}{k!} z^k N^k \mathbf{v}_i(z) \in \mathcal{O}(\mathfrak{H}) \otimes_{\mathbf{C}} V.$$

Since  $\mathbf{v}_i(z+1) = e^N \mathbf{v}_i(z)$ ,  $\mathbf{s}_i$  is  $\pi_1(\Delta^*)$ -invariant and descends to a nowhere vanishing holomorphic section of  $R^n f_*^{\circ} \mathbf{C} \otimes \mathcal{O}_{\Delta^*}$ . Since the inner product  $(\mathbf{s}_i, \mathbf{s}_j)_{L^2, z}$

is  $\pi_1(\Delta^*)$ -invariant by (2.2), it is denoted by  $(\mathbf{s}_i, \mathbf{s}_j)_{L^2, t}$ , where  $t = \exp(2\pi\sqrt{-1}z)$ . After Schmid [16, p.235], the *canonical extension* of  $R^n f_*^o \mathbf{C} \otimes \mathcal{O}_{\Delta^*}$  to  $\Delta$ , denoted by  $\mathcal{H}^n$ , is defined as the holomorphic vector bundle of rank  $\ell$  over  $\Delta$  generated by the frame fields  $\{\mathbf{s}_1, \dots, \mathbf{s}_\ell\}$  (see [17], [22] for algebro-geometric construction):

$$\mathcal{H}^n := \mathcal{O}_{\Delta} \mathbf{s}_1 \oplus \dots \oplus \mathcal{O}_{\Delta} \mathbf{s}_\ell.$$

Since  $N$  is nilpotent and has constant entries with respect to  $\{\mathbf{v}_1(z), \dots, \mathbf{v}_\ell(z)\}$ , there exists by (2.8) polynomials  $P_{ij}(z) \in \mathbf{C}[z]$  such that for all  $z \in \mathfrak{H}$

$$\mathbf{s}_i(e^{2\pi\sqrt{-1}z}) = \sum_{j=1}^{\ell} P_{ij}(z) \mathbf{v}_j(z) \quad (i = 1, \dots, \ell).$$

Since  $\{\mathbf{s}_1(e^{2\pi\sqrt{-1}z}), \dots, \mathbf{s}_\ell(e^{2\pi\sqrt{-1}z})\}$  is a basis of  $H^n(Z_{\varpi(\exp(2\pi\sqrt{-1}z))}, \mathbf{C})$ , we get

$$\det(P_{ij}(z)) \neq 0 \quad (\forall z \in \mathfrak{H}).$$

As before, for a multi-index  $I = \{i_1 < i_2 < \dots < i_r\}$ , we set

$$\mathbf{s}_I(\exp(2\pi\sqrt{-1}z)) := \mathbf{s}_{i_1}(\exp(2\pi\sqrt{-1}z)) \wedge \dots \wedge \mathbf{s}_{i_r}(\exp(2\pi\sqrt{-1}z)).$$

Then we have

$$\sum_{|I|=r} \xi_I \mathbf{s}_I(e^{2\pi\sqrt{-1}z}) = \sum_{|J|=r} \left( \sum_{|I|=r} \xi_I \begin{vmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{vmatrix} \right) \mathbf{v}_J(z),$$

where  $I = \{i_1 < \dots < i_r\}$ ,  $J = \{j_1 < \dots < j_r\}$ . We deduce from (2.7) that for all  $z \in \mathfrak{H}$  with  $a < \Re z < b$  and  $\xi = (\xi_I)_{|I|=r} \in \mathbf{C}^{\binom{\ell}{r}}$ ,

$$(2.9) \quad C_1^r(\Im z)^{-kr} \leq \frac{\|\sum_{|I|=r} \xi_I \mathbf{s}_I(e^{2\pi\sqrt{-1}z})\|_{L^2, z}^2}{\sqrt{\sum_{|J|=r} \left| \sum_{|I|=r} \xi_I \begin{vmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{vmatrix} \right|^2}} \leq C_2^r(\Im z)^{kr}.$$

We define an invertible  $\binom{\ell}{r} \times \binom{\ell}{r}$ -matrix  $\Lambda^r P(z)$  by

$$\Lambda^r P(z) := \left( \begin{vmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{vmatrix} \right)_{|I|=|J|=r} \in GL\left(\mathbf{C}^{\binom{\ell}{r}}\right).$$

Let  $\mu_{\min}(z)$  (resp.  $\mu_{\max}(z)$ ) be the smallest (resp. largest) eigenvalue of the positive-definite  $\binom{\ell}{r} \times \binom{\ell}{r}$ -matrix  $G(z) := {}^t(\Lambda^r P(z)) \overline{\Lambda^r P(z)}$ . Then we have

$$(2.10) \quad \mu_{\min}(z)^{-1} \leq \text{Tr}\{G(z)^{-1}\}, \quad \mu_{\max}(z) \leq \text{Tr} G(z).$$

For multi-indices  $I, J$  with  $|I| = |J| = r$ , let  $r_{IJ}(z) \in \mathbf{C}(z)$  be the  $(I, J)$ -entry of the  $\binom{\ell}{r} \times \binom{\ell}{r}$ -matrix  $(\Lambda^r P(z))^{-1}$ . By the definition of  $G(z)$ , we get

$$\text{Tr}\{G(z)^{-1}\} = \sum_{|I|=|J|=r} |r_{IJ}(z)|^2,$$

$$\text{Tr} G(z) = \sum_{|I|=|J|=r} \left| \det \begin{pmatrix} P_{i_1 j_1}(z) & \dots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \dots & P_{i_r j_r}(z) \end{pmatrix} \right|^2.$$



Since  $r_{IJ}(z)$  is a rational function in the variable  $z$  with  $\det(r_{IJ}(z)) \neq 0$  in  $\mathbf{C}(z)$  and since  $P_{ij}(z)$  is a polynomial in the variable  $z$  with  $\det(P_{IJ}(z)) \neq 0$  in  $\mathbf{C}[z]$ , there exist  $\mu \in \mathbf{Z}$ ,  $\nu \in \mathbf{Z}_{\geq 0}$  and constants  $C_3, C_4 > 0$  such that

$$(2.11) \quad \mathrm{Tr}\{G(z)^{-1}\} \leq C_3 |z|^{2\mu}, \quad \mathrm{Tr} G(z) \leq C_4 |z|^{2\nu}$$

for all  $z \in \mathfrak{H}$  with  $|z| \gg 1$ . Since

$$\sum_{|J|=r} \left| \sum_{|I|=r} \xi_I \begin{pmatrix} P_{i_1 j_1}(z) & \cdots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \cdots & P_{i_r j_r}(z) \end{pmatrix} \right|^2 = \|\Lambda^r P(z)\xi\|^2 = (G(z)\xi, \xi),$$

we deduce from (2.10), (2.11) that for  $z \in \mathfrak{H}$  with  $|z| \gg 1$  and  $\xi = (\xi_I)_{|I|=r} \in \mathbf{C}^{\binom{\ell}{r}}$

$$(2.12) \quad C_3^{-1} \|\xi\|^2 |z|^{-2\mu} \leq \sum_{|J|=r} \left| \sum_{|I|=r} \xi_I \begin{pmatrix} P_{i_1 j_1}(z) & \cdots & P_{i_1 j_r}(z) \\ \vdots & \ddots & \vdots \\ P_{i_r j_1}(z) & \cdots & P_{i_r j_r}(z) \end{pmatrix} \right|^2 \leq C_4 \|\xi\|^2 |z|^{2\nu}.$$

By (2.9), (2.12), we get for all  $z \in \mathfrak{H}$  with  $a < \Re z < b$ ,  $|z| \gg 1$  and  $\xi \in \mathbf{C}^{\binom{\ell}{r}}$

$$(2.13) \quad C_5 \|\xi\|^2 (\Im z)^{-(kr+2\mu)} \leq \left\| \sum_{|I|=r} \xi_I \mathbf{s}_I(e^{2\pi\sqrt{-1}z}) \right\|_{L^2, z}^2 \leq C_6 \|\xi\|^2 (\Im z)^{(kr+2\nu)},$$

where  $C_5, C_6$  are constants. Here we used the fact that  $|z|/\Im z$  is bounded from below and above by positive constants on the domain  $\{z \in \mathfrak{H}; a < \Re z < b, |z| \gg 1\}$ . Since  $\|\mathbf{s}_I(e^{2\pi\sqrt{-1}z})\|_{L^2, z}$  is  $\mathbf{Z}$ -invariant, it follows from (2.13) that for  $t \in \Delta^*$  with  $0 < |t| \ll 1$ ,

$$(2.14) \quad C_5 \|\xi\|^2 (-\log |t|)^{-(kr+2\mu)} \leq \left\| \sum_{|I|=r} \xi_I \mathbf{s}_I(t) \right\|_{L^2, t}^2 \leq C_6 \|\xi\|^2 (-\log |t|)^{(kr+2\nu)}.$$

**2.3.  $L^2$ -length of a nowhere vanishing section of  $\det \mathcal{F}^p$ .** On  $\mathcal{H}^n$ , we have the Hodge filtration

$$0 \subset \mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \cdots \subset \mathcal{F}^1 \subset \mathcal{F}^0 = \mathcal{H}^n,$$

where  $\mathcal{F}^p$  is a holomorphic subbundle of  $\mathcal{H}^n$  satisfying  $\mathcal{F}_t^p = \bigoplus_{k \geq p} H^{k, n-k}(Z_t)$  for  $t \in \Delta^*$  and

$$\mathcal{F}^p / \mathcal{F}^{p+1} \cong R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log Z_0).$$

Here  $\Omega_{\mathcal{Z}/\Delta}^p(\log Z_0) := \Lambda^p \Omega_{\mathcal{Z}/\Delta}^1(\log Z_0)$  and  $\Omega_{\mathcal{Z}/\Delta}^1(\log Z_0) := \Omega_{\mathcal{Z}}^1(\log Z_0) / \mathcal{O}_{\mathcal{Z}} f^*(dt/t)$ .

On the open subset of  $\mathcal{Z}$  on which  $f(z) = z_1 \cdots z_k$ ,  $\Omega_{\mathcal{Z}}^1(\log Z_0)$  is given by

$$\Omega_{\mathcal{Z}}^1(\log Z_0) = \mathcal{O}_{\mathcal{Z}}(dz_1/z_1) + \cdots + \mathcal{O}_{\mathcal{Z}}(dz_k/z_k) + \mathcal{O}_{\mathcal{Z}} dz_{k+1} + \cdots + \mathcal{O}_{\mathcal{Z}} dz_n.$$

See [17], [22] for algebro-geometric account of the Hodge filtrations. In what follows, we often write  $R^q f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)$  for  $R^q f_* \Omega_{\mathcal{Z}/\Delta}^p(\log Z_0)$ .

Set  $\ell_p := \mathrm{rk} \mathcal{F}^p$ . There exist holomorphic sections  $\varphi_1, \dots, \varphi_{\ell} \in \Gamma(\Delta, \mathcal{H}^n)$  with

$$\mathcal{F}^p = \mathcal{O}_{\Delta} \varphi_1 \oplus \cdots \oplus \mathcal{O}_{\Delta} \varphi_{\ell_p} \quad (p = n, n-1, \dots, 1, 0).$$

Since  $\{\mathbf{s}_1, \dots, \mathbf{s}_{\ell}\}$  is a basis of  $\mathcal{H}^n$  as an  $\mathcal{O}_{\Delta}$ -module, there exist holomorphic functions  $a_{\alpha i}(t) \in \mathcal{O}(\Delta)$ ,  $1 \leq i, \alpha \leq \ell$  such that

$$\varphi_{\alpha}(t) = \sum_{i=1}^{\ell} a_{\alpha i}(t) \mathbf{s}_i(t) \quad (\alpha = 1, \dots, \ell), \quad \det(a_{\alpha i}(0))_{1 \leq \alpha, i \leq \ell} \neq 0.$$

**Proposition 2.2.** *There exist constants  $C \in \mathbf{R}_{\geq 0}$  and  $C' \in \mathbf{R}_{>0}$  such that for all  $t \in \Delta^*$  with  $|t| \ll 1$  and  $1 \leq m \leq \ell$ ,*

$$\left| \log \|\varphi_1(t) \wedge \cdots \wedge \varphi_m(t)\|_{L^2, t} \right| \leq C' + C \log(-\log |t|).$$

*Proof.* Since

$$\varphi_1(t) \wedge \cdots \wedge \varphi_m(t) = \sum_{|J|=m} \begin{vmatrix} a_{1j_1}(t) & \cdots & a_{1j_m}(t) \\ \vdots & \ddots & \vdots \\ a_{mj_1}(t) & \cdots & a_{mj_m}(t) \end{vmatrix} \mathbf{s}_J(t),$$

there exist by (2.14) constants  $k_1, k_2 \geq 0$  such that for all  $t \in \Delta^*$  with  $0 < |t| \ll 1$  (2.15)

$$C_5 (-\log |t|)^{-k_1} \leq \frac{\|\varphi_1(t) \wedge \cdots \wedge \varphi_m(t)\|_{L^2, t}^2}{\sum_{|J|=m} \left| \det \begin{pmatrix} a_{1j_1}(t) & \cdots & a_{1j_m}(t) \\ \vdots & \ddots & \vdots \\ a_{mj_1}(t) & \cdots & a_{mj_m}(t) \end{pmatrix} \right|^2} \leq C_6 (-\log |t|)^{k_2}.$$

By (2.15), it suffices to prove that

$$(2.16) \quad \sum_{|J|=m} \left| \det \begin{pmatrix} a_{1j_1}(0) & \cdots & a_{1j_m}(0) \\ \vdots & \ddots & \vdots \\ a_{mj_1}(0) & \cdots & a_{mj_m}(0) \end{pmatrix} \right|^2 \neq 0.$$

Set

$$A := \begin{pmatrix} a_{11}(0) & \cdots & a_{1\ell}(0) \\ \vdots & \ddots & \vdots \\ a_{\ell 1}(0) & \cdots & a_{\ell \ell}(0) \end{pmatrix} \in GL(\mathbf{C}^\ell).$$

Then the endomorphism on  $\Lambda^m \mathbf{C}^\ell$  induced by  $A$  is given by the  $\binom{\ell}{m} \times \binom{\ell}{m}$ -matrix

$$\Lambda^m A := \left( \begin{vmatrix} a_{i_1 j_1}(0) & \cdots & a_{i_1 j_m}(0) \\ \vdots & \ddots & \vdots \\ a_{i_m j_1}(0) & \cdots & a_{i_m j_m}(0) \end{vmatrix} \right)_{|I|=|J|=m}.$$

Since  $\det A \neq 0$ ,  $\Lambda^m A \in \text{End}(\Lambda^m \mathbf{C}^\ell)$  is invertible. In particular, the row vector of  $\Lambda^m A$  corresponding to the multi-index  $I = \{1, 2, \dots, m\}$  is non-zero, which implies (2.16). This proves the result.  $\square$

Since  $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)|_{\Delta^*} = R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p|_{\Delta^*}$  is identified with the holomorphic vector bundle over  $\Delta^*$  with fiber  $H^{n-p}(Z_t, \Omega_{Z_t}^p)$  over  $t \in \Delta^*$ ,  $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)|_{\Delta^*}$  is equipped with the  $L^2$ -Hermitian metric by identifying  $H^{n-p}(Z_t, \Omega_{Z_t}^p)$  with the corresponding vector space of harmonic forms of bidegree  $(p, n-p)$ .

Under the canonical identification  $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log) = \mathcal{F}^p / \mathcal{F}^{p+1}$ , the  $L^2$ -metric on  $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)|_{\Delta^*}$  is identified with the quotient metric on  $\mathcal{F}^p / \mathcal{F}^{p+1}$  induced by the  $L^2$ -metric on  $\mathcal{H}^p|_{\Delta^*}$ . Hence we have an isometry of holomorphic line bundles equipped with singular Hermitian metrics

$$(2.17) \quad \left( \det R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log), \|\cdot\|_{L^2} \right) \cong (\det \mathcal{F}^p, \|\cdot\|_{L^2}) \otimes (\det \mathcal{F}^{p+1}, \|\cdot\|_{L^2})^\vee.$$

**Corollary 2.3.** *Let  $e_1(t), \dots, e_{n_p}(t) \in \Gamma(\Delta, R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log))$  be a basis of the free  $\mathcal{O}_\Delta$ -module  $R^{n-p} f_* \Omega_{\mathcal{Z}/\Delta}^p(\log)$ , where  $n_p := h^{p, n-p}(Z_t)$ ,  $t \neq 0$ . Then there exist constants  $C \geq 0$  and  $C' > 0$  such that for all  $t \in \Delta^*$  with  $|t| \ll 1$*

$$\left| \log \|e_1(t) \wedge \dots \wedge e_{n_p}(t)\|_{L^2, t} \right| \leq C' + C \log(-\log |t|).$$

*Proof.* There exist nowhere vanishing holomorphic sections  $\mathbf{f}_p(t) \in \Gamma(\Delta, \det \mathcal{F}^p)$  and  $\mathbf{f}_{p+1}(t) \in \Gamma(\Delta, \det \mathcal{F}^{p+1})$  such that

$$e_1(t) \wedge \dots \wedge e_{n_p}(t) = \mathbf{f}_p(t) \otimes \mathbf{f}_{p+1}(t)^{-1}$$

under the identification (2.17). Since

$$\|e_1(t) \wedge \dots \wedge e_{n_p}(t)\|_{L^2, t} = \|\mathbf{f}_p(t)\|_{L^2, t} \cdot \|\mathbf{f}_{p+1}(t)\|_{L^2, t}^{-1}$$

the result follows from Proposition 2.2 applied to the nowhere vanishing sections  $\mathbf{f}_p(t)$  and  $\mathbf{f}_{p+1}(t)$ .  $\square$

### 3. SINGULARITY OF BCOV INVARIANTS FOR SEMI-STABLE DEGENERATIONS

**Set up.** Throughout this section except Section 3.9, we assume the following:

- (1) There exist a smooth projective fourfold  $\mathcal{X}$ , a compact Riemann surface  $C$ , an embedding  $\Delta \subset C$ , and a surjective holomorphic map  $f: \mathcal{X} \rightarrow C$  such that  $f: \mathcal{X}|_\Delta \rightarrow \Delta$  is a semi-stable degeneration. In particular,  $f^{-1}(0)$  is a reduced normal crossing divisor of  $\mathcal{X}$ .
- (2) The regular fibers of  $f: \mathcal{X} \rightarrow C$  are Calabi-Yau threefolds.

Set  $X_t := f^{-1}(t)$  for  $t \in C$ . Then  $X_t$  is a Calabi-Yau threefold for all  $t \in \Delta^*$ . In this section, we determine the asymptotic behavior of the function on  $\Delta^*$

$$t \mapsto \log \tau_{\text{BCOV}}(X_t) \quad (t \rightarrow 0).$$

Let  $\mathcal{L}$  be an ample line bundle on  $\mathcal{X}$ . For all  $t \in \Delta^*$ ,  $c_1(\mathcal{L}_t)$  is a Kähler class on  $X_t$ , so that every  $H^{p,q}(X_t, \mathbf{C})$  is endowed with the  $L^2$ -Hermitian structure with respect to the Kähler class  $c_1(\mathcal{L}_t)$ . When  $p+q=3$ ,  $H^{p,q}(X_t, \mathbf{C})$  consists of primitive cohomology classes and the  $L^2$ -Hermitian structure on  $H^{p,q}(X_t, \mathbf{C})$  with respect to  $c_1(\mathcal{L}_t)$  is given by (2.1).

Let  $\Sigma_f := \{x \in \mathcal{X}; df_x = 0\}$  be the critical locus of  $f$  and let  $\Delta_f := f(\Sigma_f) \subset C$  be the discriminant locus of  $f: \mathcal{X} \rightarrow C$ . Then  $\Omega_{\mathcal{X}/C}^p$  is a holomorphic vector bundle over  $\mathcal{X} \setminus \Sigma_f$ , where  $\Omega_{\mathcal{X}/C}^p := \Lambda^p \Omega_{\mathcal{X}/C}^1$  and  $\Omega_{\mathcal{X}/C}^1 := \Omega_{\mathcal{X}}^1 / f^* \Omega_C^1$ . Since possible extension of  $\Omega_{\mathcal{X}/C}^p$  to a coherent sheaf on  $\mathcal{X}$  is not unique in general, we regard  $\Omega_{\mathcal{X}/C}^p$  as a locally free sheaf on  $\mathcal{X} \setminus \Sigma_f$  rather than a coherent sheaf on  $\mathcal{X}$ .

Recall that the determinant of the cohomologies of  $\Omega_{\mathcal{X}/C}^p$  is the holomorphic line bundle on  $C \setminus \Delta_f$  defined as

$$\lambda(\Omega_{\mathcal{X}/C}^p) := \bigotimes_{q \geq 0} (\det R^q f_* \Omega_{\mathcal{X}/C}^p)^{(-1)^q}.$$

Since  $X_t$ ,  $t \in C \setminus \Delta_f$ , is equipped with the Kähler class  $c_1(\mathcal{L}_t)$ ,  $H^q(X_t, \Omega_{X_t}^p)$  is equipped with the  $L^2$  Hermitian metric by identifying it with the corresponding vector space of harmonic forms. In this way, for all  $p, q \geq 0$ ,  $R^q f_* \Omega_{\mathcal{X}/C}^p$  is a holomorphic vector bundle over  $C \setminus \Delta_f$  equipped with the  $L^2$  Hermitian metric. (This Hermitian metric coincides with the one considered in Section 2.3 when  $p+q =$

3.) Hence  $\lambda(\Omega_{\mathcal{X}/C}^p)$  is a holomorphic Hermitian line bundle on  $C \setminus \Delta_f$  equipped with the  $L^2$  metric  $\|\cdot\|_{L^2}$ .

We fix a Kähler metric  $g^{\mathcal{X}}$  on  $\mathcal{X}$  with Kähler class  $c_1(\mathcal{L})$ . For  $t \in C \setminus \Delta_f$ , we set  $g_t := g^{\mathcal{X}}|_{X_t}$ . Then  $\{(X_t, g_t)\}_{t \in C \setminus \Delta_f}$  is a family of compact Kähler manifolds with constant Kähler class  $c_1(\mathcal{L}_t) = c_1(\mathcal{L})|_{X_t}$ . As in Section 1.1, we have analytic torsion  $\tau(X_t, \Omega_{X_t}^p)$  for all  $t \in C \setminus \Delta_f$ . By Quillen [14] and Bismut-Gillet-Soulé [4], the *Quillen metric* on  $\lambda(\Omega_{\mathcal{X}/C}^p)$  is defined as

$$\|\cdot\|_Q^2(t) := \tau(X_t, \Omega_{X_t}^p) \cdot \|\cdot\|_{L^2, t}^2, \quad t \in C \setminus \Delta_f.$$

The curvature and anomaly formulae for Quillen metrics were obtained by Bismut-Gillet-Soulé [4]. As an application of the Bismut-Lebeau embedding formula [5], the singularity of Quillen metric as  $t \rightarrow 0 \in \Delta_f$  was determined in [7], which we shall recall in Section 3.4.

### 3.1. Asymptotic behavior of the $L^2$ -metric on $\lambda(\mathcal{O}_{\mathcal{X}})|_{\Delta}$ .

**Proposition 3.1.** *Let  $\varsigma_0 \in \Gamma(\Delta, \lambda(\mathcal{O}_{\mathcal{X}})|_{\Delta})$  be a nowhere vanishing holomorphic section. Then the following holds as  $t \rightarrow 0$ :*

$$\log \|\varsigma_0(t)\|_{L^2}^2 = O(\log(-\log |t|)).$$

*Proof.* Since  $X_t$  is a Calabi-Yau threefold for all  $t \in \Delta^*$ , we have  $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_C \cdot 1$  and  $R^3 f_*\mathcal{O}_{\mathcal{X}}|_{\Delta} = \mathcal{O}_{\Delta} \cdot e$ , where  $e(t) \in \Gamma(\Delta, R^3 f_*\mathcal{O}_{\mathcal{X}})$  is a nowhere vanishing section. By the definition of the  $L^2$ -metric on  $H^0(X_t, \mathbf{C})$ , we see that  $\|1\|_{L^2}^2 = \deg(L_t)/(2\pi)^3$  is a constant function on  $\Delta$ . By Corollary 2.3 in the case  $n = 3$ ,  $q = 0$ , we have

$$\log \|e(t)\|_{L^2}^2 = O(\log(-\log |t|)) \quad (t \rightarrow 0).$$

Since  $\sigma_0 = 1 \otimes e^{\vee}$ , we get the result.  $\square$

### 3.2. Asymptotic behavior of the $L^2$ -metric on $\lambda(\Omega_{\mathcal{X}/C}^1)|_{\Delta}$ . We define

$$\Omega_{\mathcal{X}/C}^1 := \Omega_{\mathcal{X}}^1 / f^* \Omega_C^1.$$

Since  $\Omega_{\mathcal{X}}^1 \subset \Omega_{\mathcal{X}}^1(\log)$  and  $\Omega_C^1 \subset \Omega_C^1(\log)$ , we have the natural inclusion of sheaves

$$\Omega_{\mathcal{X}/C}^1 \subset \Omega_{\mathcal{X}/C}^1(\log)$$

and we set

$$\mathcal{Q} := \Omega_{\mathcal{X}/C}^1(\log) / \Omega_{\mathcal{X}/C}^1.$$

Then  $\mathcal{Q}|_{f^{-1}(\Delta)}$  is a coherent sheaf on  $\mathcal{X}|_{\Delta}$  supported on  $\text{Sing } X_0$ . The short exact sequence of coherent sheaves on  $\mathcal{X}$

$$0 \longrightarrow \Omega_{\mathcal{X}/C}^1 \longrightarrow \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow \mathcal{Q} \longrightarrow 0$$

induces the long exact sequence of direct image sheaves on  $C$

$$(3.1) \quad \begin{aligned} & \longrightarrow R^{q-1} f_* \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow R^{q-1} f_* \mathcal{Q} \longrightarrow R^q f_* \Omega_{\mathcal{X}/C}^1 \longrightarrow R^q f_* \Omega_{\mathcal{X}/C}^1(\log) \longrightarrow R^q f_* \mathcal{Q} \longrightarrow \end{aligned}$$

Following [7, proof of Prop. 9.5], set

$$M_q := (R^q f_* \Omega_{\mathcal{X}/C}^1)_{\text{tors}}|_{\Delta}, \quad N_q := R^q f_* \Omega_{\mathcal{X}/C}^1(\log) / R^q f_* \Omega_{\mathcal{X}/C}^1|_{\Delta}.$$

Since  $R^q f_* \Omega_{\mathcal{X}/C}^1(\log)$  is a locally free sheaf on  $C$ , we deduce from (3.1) the isomorphism of  $\mathcal{O}_{\Delta}$ -modules

$$(3.2) \quad R^q f_* \mathcal{Q}|_{\Delta} \cong N_q \oplus M_{q+1}.$$

Set

$$\chi(Rf_*\mathcal{Q}|_\Delta) := \sum_q (-1)^q \dim_{\mathbf{C}}(R^q f_*\mathcal{Q})_0.$$

By (3.2), we get

$$(3.3) \quad \chi(Rf_*\mathcal{Q}|_\Delta) = \sum_q (-1)^q (\dim_{\mathbf{C}}(N_q)_0 - \dim_{\mathbf{C}}(M_q)_0).$$

Since  $\mathcal{Q}|_{f^{-1}(\Delta)}$  depends only on the function germ of  $f$  near  $\Sigma_f$ , so is  $\chi(Rf_*\mathcal{Q}|_\Delta)$ .

**Proposition 3.2.** *Let  $\varsigma_1 \in \Gamma(\Delta, \lambda(\Omega_{\mathcal{X}/C}^1))$  be a nowhere vanishing holomorphic section. Then the following holds as  $t \rightarrow 0$ :*

$$\log \|\varsigma_1(t)\|_{\lambda(\Omega_{\mathcal{X}/C}^1), L^2}^2 = \chi(Rf_*\mathcal{Q}|_\Delta) \log |t|^2 + O(\log(-\log |t|)).$$

*Proof.* Let  $e_1(t), \dots, e_{h^{1,q}}(t) \in \Gamma(\Delta, R^q f_*\Omega_{\mathcal{X}/C}^1(\log))$  be a basis of  $R^q f_*\Omega_{\mathcal{X}/C}^1(\log)|_\Delta$  as a free  $\mathcal{O}_\Delta$ -module. By [7, Prop. 9.4], there exists  $\delta_q \in \mathbf{R}$  such that

$$\log \|e_1(t) \wedge \dots \wedge e_{h^{1,q}}(t)\|_{L^2}^2 = \delta_q \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0).$$

It follows from [7, Eq. (9.15)] and (3.3) that as  $t \rightarrow 0$

$$(3.4) \quad \log \|\varsigma_1(t)\|_{\lambda(\Omega_{\mathcal{X}/C}^1), L^2}^2 = \{\chi(Rf_*\mathcal{Q}|_\Delta) + \sum_q (-1)^q \delta_q\} \log |t|^2 + O(\log(-\log |t|)).$$

By (3.4), it suffices to prove  $\delta_q = 0$  for all  $q \geq 0$ . The vanishing  $\delta_0 = \delta_1 = \delta_3 = 0$  was already proved in [7, proof of Prop. 9.4]. The vanishing  $\delta_2 = 0$  follows from Corollary 2.3. This completes the proof.  $\square$

**3.3. The Kähler extension of  $\lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*}$ .** Following [7], we recall an extension of  $\lambda(\Omega_{\mathcal{X}/C}^p)$  from  $\Delta^*$  to  $\Delta$ , which we call the Kähler extension and which is distinct from  $\lambda(\Omega_{\mathcal{X}/C}^p(\log))$  in general. For  $p \geq 0$ , set

$$\Omega_{\mathcal{X}/C}^p := \Lambda^p \Omega_{\mathcal{X}/C}^1.$$

Then  $\Omega_{\mathcal{X}/C}^p$  is a coherent sheaf on  $\mathcal{X}$ , which is locally free on  $\mathcal{X} \setminus \Sigma_f$ . Following [7, Sect. 5], we recall the Kähler extension of  $\lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*}$ . On  $\mathcal{X}|_{\Delta^*}$ , we have the following exact sequence of holomorphic vector bundles:

$$0 \longrightarrow f^*\Omega_C^1 \longrightarrow \Omega_{\mathcal{X}}^1 \longrightarrow \Omega_{\mathcal{X}/C}^1 \longrightarrow 0.$$

Since  $\text{rk } f^*\Omega_C^1 = 1$ , this short exact sequence induces the following exact sequence of holomorphic vector bundles on  $\mathcal{X}|_{\Delta^*}$ :

$$(3.5) \quad 0 \longrightarrow \mathcal{E}_{\mathcal{X}/C}^p \longrightarrow \Omega_{\mathcal{X}/C}^p \longrightarrow 0,$$

where  $\mathcal{E}_{\mathcal{X}/C}^p$  is the complex of holomorphic vector bundles over  $\mathcal{X}$  given by

$$\mathcal{E}_{\mathcal{X}/C}^p: (f^*\Omega_C^1)^{\otimes p} \rightarrow \Omega_{\mathcal{X}}^1 \otimes (f^*\Omega_C^1)^{\otimes p-1} \rightarrow \dots \rightarrow \Omega_{\mathcal{X}}^{p-1} \otimes f^*\Omega_C^1 \rightarrow \Omega_{\mathcal{X}}^p$$

and the map  $\Omega_{\mathcal{X}}^p \rightarrow \Omega_{\mathcal{X}/C}^p$  is given by the canonical quotient map. Here, if  $\theta$  is a local generator of  $\Omega_C^1$ , then the map  $\Omega_{\mathcal{X}}^i \otimes (f^*\Omega_C^1)^{\otimes (p-i)} \rightarrow \Omega_{\mathcal{X}}^{i+1} \otimes (f^*\Omega_C^1)^{\otimes (p-i-1)}$  is given by  $\omega \otimes (f^*\theta)^{\otimes (p-i)} \mapsto (\omega \wedge f^*\theta) \otimes (f^*\theta)^{\otimes (p-i-1)}$  for  $\omega \in \Omega_{\mathcal{X}}^{p-i}$ .

**Definition 3.3.** The *Kähler extension* of  $\lambda(\Omega_{\mathcal{X}/C}^p)$  is the holomorphic line bundle over  $C$  defined as

$$\lambda(\mathcal{E}_{\mathcal{X}/C}^p) := \bigotimes_{i=0}^p \lambda \left( \Omega_{\mathcal{X}}^{p-i} \otimes (f^* \Omega_C^1)^{\otimes i} \right)^{(-1)^i}.$$

By the exactness of (3.5) on  $\mathcal{X}|_{\Delta^*}$ , we have the canonical isomorphism of holomorphic line bundles over  $\Delta^*$ :

$$(3.6) \quad \lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*} \cong \lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta^*}.$$

When  $p = 0, 1$ , the canonical isomorphism (3.6) extends to an isomorphism of holomorphic line bundles over  $C$ :

$$(3.7) \quad \lambda(\mathcal{O}_{\mathcal{X}}) = \lambda(\mathcal{E}_{\mathcal{X}/C}^0), \quad \lambda(\Omega_{\mathcal{X}/C}^1) \cong \lambda(\mathcal{E}_{\mathcal{X}/C}^1).$$

Via the canonical isomorphism (3.6), the  $L^2$ -metric on  $\lambda(\Omega_{\mathcal{X}/C}^p)|_{\Delta^*}$  induces a Hermitian metric on  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta^*}$ , which is denoted by  $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}$ . Notice that  $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}$  does *not* coincide with the Hermitian metric on  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)$  defined as the product of the  $L^2$ -metrics on  $\lambda(\Omega_{\mathcal{X}}^{p-i} \otimes (f^* \Omega_C^1)^{\otimes i})$ .

**3.4. Asymptotic behavior of the Quillen metric on  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta}$ .** Via the canonical isomorphism (3.6), the line bundle  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_{\Delta^*}$  is endowed with the Quillen metric  $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}$ . Following [7, Sect. 5], we recall the singularity of  $\|\cdot\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}$  at  $t = 0$ . For this, we introduce some tautological vector bundles over the projective bundle  $\mathbf{P}(T\mathcal{X})^\vee$ .

Let  $\mathbf{P}(T\mathcal{X})^\vee$  be the projective bundle over  $\mathcal{X}$  with projection  $\Pi: \mathbf{P}(T\mathcal{X})^\vee \rightarrow \mathcal{X}$  such that  $\Pi^{-1}(x) = \mathbf{P}(T_x\mathcal{X})^\vee$ . Here, for a complex vector space  $V$ ,  $\mathbf{P}(V)^\vee$  denotes the projective space of hyperplanes of  $V$  passing through the origin. Then we have the canonical isomorphism  $\mathbf{P}(T\mathcal{X})^\vee \cong \mathbf{P}(\Omega_{\mathcal{X}}^1)$ . We define the *Gauss map*  $\mu: \mathcal{X} \setminus \Sigma_f \rightarrow \mathbf{P}(T\mathcal{X})^\vee$  by

$$\mu(x) := [T_x X_{f(x)}],$$

where  $[T_x X_{f(x)}] \in \mathbf{P}(T_x\mathcal{X})^\vee$  is the point corresponding to the hyperplane  $T_x X_{f(x)} \subset T_x\mathcal{X}$ . Then  $\mu$  is a meromorphic map from  $\mathcal{X}$  to  $\mathbf{P}(T\mathcal{X})^\vee$ . Let

$$\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$$

be a resolution of the indeterminacy of  $\mu$ . Namely, there exists a birational holomorphic map  $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  inducing an isomorphism between  $\tilde{\mathcal{X}} \setminus \sigma^{-1}(\Sigma_f)$  and  $\mathcal{X} \setminus \Sigma_f$  such that the composite morphism

$$\tilde{\mu} := \mu \circ \sigma$$

extends to a holomorphic map from  $\tilde{\mathcal{X}}$  to  $\mathbf{P}(T\mathcal{X})^\vee$ . We set

$$\text{Exc}(\sigma) := \sigma^{-1}(\Sigma_f).$$

Without loss of generality, we may and will assume that  $\text{Exc}(\sigma)$  is a normal crossing divisor of  $\tilde{\mathcal{X}}$ .

*Remark 3.4.* Since  $f: \mathcal{X}|_{\Delta} \rightarrow \Delta$  is a semi-stable degeneration, for any  $p \in X_0$ , there is a system of local coordinates  $(z_0, z_1, z_2, z_3)$  of  $\mathcal{X}$  centered at  $p$  such that

$$f(z) = z_0 \cdots z_k \quad (k \leq 3).$$

Near  $p$ , the Gauss map  $\mu$  is expressed as the following explicit meromorphic map

$$\mu(z) = \left( \frac{1}{z_0} : \cdots : \frac{1}{z_k} : 0 : \cdots : 0 \right).$$

One can resolve the indeterminacy of  $\mu$  in the canonical way as follows.

Let  $X_0 = E_1 + \cdots + E_m$  be the irreducible decomposition. For  $i_1 < \cdots < i_k$ , we set  $E_{i_1 \dots i_k} := E_{i_1} \cap \cdots \cap E_{i_k}$ . Then  $E_{i_1 \dots i_k}$  is a (possibly disconnected) submanifold of  $\mathcal{X}$  and  $E_{i_1 \dots i_k} = \emptyset$  for  $k \geq 5$ . The indeterminacy locus of  $\mu$  is given by  $\bigcup_{i < j} E_{ij}$ . Let  $\sigma^{(1)}: \mathcal{X}^{(1)} \rightarrow \mathcal{X}$  be the blowing-up of  $\bigcup_{i < j < k < l} E_{ijkl}$  and set  $\mu^{(1)} := \mu \circ \sigma^{(1)}$ . Let  $E_i^{(1)}$  be the proper transform of  $E_i$  and set  $E_{i_1 \dots i_k}^{(1)} := E_{i_1}^{(1)} \cap \cdots \cap E_{i_k}^{(1)}$  for  $i_1 < \cdots < i_k$ . Then  $E_{i_1 \dots i_k}^{(1)} = \emptyset$  for  $k \geq 4$  and  $E_{i_1 \dots i_k}^{(1)}$  is the proper transform of  $E_{i_1 \dots i_k}$  for  $k \leq 3$ . The indeterminacy locus of  $\mu^{(1)}$  is given by  $\bigcup_{i < j} E_{ij}^{(1)}$ . Let  $\sigma^{(2)}: \mathcal{X}^{(2)} \rightarrow \mathcal{X}^{(1)}$  be the blowing-up of  $\bigcup_{i < j < k} E_{ijk}^{(1)}$  and set  $\mu^{(2)} := \mu^{(1)} \circ \sigma^{(2)}$ . Let  $E_i^{(2)}$  be the proper transform of  $E_i^{(1)}$  and set  $E_{i_1 \dots i_k}^{(2)} := E_{i_1}^{(2)} \cap \cdots \cap E_{i_k}^{(2)}$  for  $i_1 < \cdots < i_k$ . Then  $E_{i_1 \dots i_k}^{(2)} = \emptyset$  for  $k \geq 3$  and  $E_{i_1 \dots i_k}^{(2)}$  is the proper transform of  $E_{i_1 \dots i_k}^{(1)}$  for  $k \leq 2$ . The indeterminacy locus of  $\mu^{(2)}$  is given by  $\bigcup_{i < j} E_{ij}^{(2)}$ . Finally, let  $\sigma^{(3)}: \mathcal{X}^{(3)} \rightarrow \mathcal{X}^{(2)}$  be the blowing-up of  $\bigcup_{i < j} E_{ij}^{(2)}$  and set  $\mu^{(3)} := \mu^{(2)} \circ \sigma^{(3)}$ . Then  $\mu^{(3)}: \mathcal{X}^{(3)} \rightarrow \mathbf{P}(T\mathcal{X})^\vee$  is regular. Setting  $\tilde{\mathcal{X}} := \mathcal{X}^{(3)}$ ,  $\sigma := \sigma^{(1)} \circ \sigma^{(2)} \circ \sigma^{(3)}$  and  $\tilde{\mu} := \mu \circ \sigma$ , we get a resolution of the indeterminacy of  $\mu: \mathcal{X} \dashrightarrow \mathbf{P}(T\mathcal{X})^\vee$ .

Let  $U$  be the universal hyperplane bundle over  $\mathbf{P}(T\mathcal{X})^\vee$  and let  $H$  be the universal quotient line bundle over  $\mathbf{P}(T\mathcal{X})^\vee$ . Then we have the following exact sequence of holomorphic vector bundles over  $\mathbf{P}(T\mathcal{X})^\vee$ :

$$0 \longrightarrow U \longrightarrow \Pi^* T\mathcal{X} \longrightarrow H \longrightarrow 0.$$

After [7, Th. 5.4], we introduce the rational number  $a_p(f, \Sigma_f) \in \mathbf{Q}$  by

$$a_p(f, \Sigma_f) := \sum_{j=0}^p (-1)^{p-j} \int_{\text{Exc}(\tilde{\sigma})} \tilde{\mu}^* \left\{ \text{Td}(U) \frac{\text{Td}(c_1(H)) - e^{-(p-j)c_1(H)}}{c_1(H)} \right\} \sigma^* \text{ch}(\Omega_{\mathcal{X}}^j).$$

Then  $a_p(f, \Sigma_f)$  is determined by the function germ of  $f: \mathcal{X}|_\Delta \rightarrow \Delta$  near  $\Sigma_f$ .

**Theorem 3.5.** *Let  $0 \leq p \leq 3$  and let  $\varsigma_p$  be a nowhere vanishing holomorphic section of  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$ . Then the following holds as  $t \rightarrow 0$ :*

$$\log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}^2 = a_p(f, \Sigma_f) \log |t|^2 + O(1).$$

*Proof.* See [7, Th. 5.4]. □

**3.5. Asymptotic behavior of the  $L^2$ -metric on  $\lambda(\Omega_{\mathcal{X}/C}^p)|_\Delta$ : the cases  $p = 2, 3$ .** Following [7, Th. 8.1 and Prop. 9.6], we determine the singularity of the  $L^2$ -metric on  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$  for the remaining cases  $p = 2, 3$ .

**Proposition 3.6.** *Let  $\varsigma_p$  be a nowhere vanishing holomorphic section of  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$ .*

(1) *When  $p = 2$ , the following holds as  $t \rightarrow 0$ :*

$$\begin{aligned} \log \|\varsigma_2(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^2), L^2}^2 &= \{a_2(f, \Sigma_f) - a_1(f, \Sigma_f) + \chi(Rf_* \mathcal{Q}|_\Delta)\} \log |t|^2 \\ &\quad + O(\log(-\log |t|)). \end{aligned}$$

(2) When  $p = 3$ , the following holds as  $t \rightarrow 0$ :

$$\log \|\varsigma_3(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^3), L^2}^2 = \{a_3(f, \Sigma_f) - a_0(f, \Sigma_f)\} \log |t|^2 + O(\log(-\log |t|)).$$

*Proof.* Let  $0 \leq p \leq 3$ . Then we have

$$\begin{aligned} & \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}^2 - \log \|\varsigma_{3-p}(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p}), L^2}^2 \\ &= \log \|\varsigma_p(t) \otimes \varsigma_{3-p}(t)^\vee\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p) \otimes \lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p})^\vee, L^2}^2 \\ (3.8) \quad &= \log \|\varsigma_p(t) \otimes \varsigma_{3-p}(t)^\vee\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p) \otimes \lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p})^\vee, Q}^2 \\ &= \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}^2 - \log \|\varsigma_{3-p}(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^{3-p}), Q}^2 \\ &= \{a_p(f, \Sigma_f) - a_{3-p}(f, \Sigma_f)\} \log |t|^2 + O(1), \end{aligned}$$

where the second equality follows from [7, Eq. (8.4)] and the last equality follows from Theorem 3.5. The result for  $p = 2$  follows from (3.8) and Proposition 3.2. The result for  $p = 3$  follows from (3.8) and Proposition 3.1.  $\square$

**3.6. Asymptotic behavior of BCOV torsion.** Following [7, Th. 8.2 and Sect. 9.2], we determine the singularity of  $T_{\text{BCOV}}(X_t, g_t)$  as  $t \rightarrow 0$ .

**Theorem 3.7.** *The following holds as  $t \rightarrow 0$ :*

$$\begin{aligned} \log T_{\text{BCOV}}(X_t, g_t) &= \{-3a_0(f, \Sigma_f) + 2a_1(f, \Sigma_f) - \chi(Rf_* \mathcal{Q}|_\Delta)\} \log |t|^2 \\ &\quad + O(\log(-\log |t|)). \end{aligned}$$

*Proof.* For simplicity, write  $a_p$  for  $a_p(f, \Sigma_f)$ . Let  $\varsigma_p$  be a nowhere vanishing holomorphic section of  $\lambda(\mathcal{E}_{\mathcal{X}/C}^p)|_\Delta$ . By Theorem 3.5, we get

$$(3.9) \quad \sum_{p=0}^3 (-1)^p p \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}^2 = \left( \sum_{p=0}^3 (-1)^p p a_p \right) \log |t|^2 + O(1).$$

By Propositions 3.1, 3.2, 3.6, we get

$$\begin{aligned} (3.10) \quad & \sum_{p=0}^3 (-1)^p p \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}^2 = \{3a_0 - 2a_1 + 2a_2 - 3a_3 + \chi(Rf_* \mathcal{Q}|_\Delta)\} \log |t|^2 \\ & \quad + O(\log(-\log |t|)). \end{aligned}$$

By the definition of Quillen metrics, we have

$$(3.11) \quad \log T_{\text{BCOV}}(X_t, g_t) = \sum_{p=0}^3 (-1)^p p \left\{ \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), Q}^2 - \log \|\varsigma_p(t)\|_{\lambda(\mathcal{E}_{\mathcal{X}/C}^p), L^2}^2 \right\}.$$

Substituting (3.9), (3.10) into (3.11), we get the result.  $\square$

**3.7. Asymptotic behavior of the Bott-Chern term.** Following [7, Prop. 7.9], we determine the singularity of  $A(X_t, g_t)$  as  $t \rightarrow 0$ .

Let  $\Xi \in \Gamma(\mathcal{X}|_\Delta, K_{\mathcal{X}})$  be a canonical form on  $\mathcal{X}|_\Delta$  such that (cf. [7, Lemma 7.7])

$$(3.12) \quad \text{div}(\Xi) \subset X_0.$$



Let  $\omega_{X_t}$  be the dualizing sheaf of  $X_t$ . Then  $\omega_{X_t} \cong K_{\mathcal{X}/C}|_{X_t}$  for all  $t \in \Delta$ . We define  $\eta_t \in H^0(X_t, \omega_{X_t})$  as the canonical form on  $X_t$  such that

$$\Xi|_{X_t} = \eta_t \wedge df.$$

Let  $\eta_{\mathcal{X}/\Delta} \in \Gamma(\Delta, f_* K_{\mathcal{X}/C})$  be the section defined by  $\eta(t) = \eta_t$  for all  $t \in \Delta$ . Since the family  $f: \mathcal{X}|_{\Delta} \rightarrow \Delta$  is a semi-stable degeneration,  $\eta_{\mathcal{X}/\Delta}$  is regarded as a holomorphic section of the Hodge bundle  $\mathcal{F}^3 \subset \mathcal{H}^3$ . If  $\Xi$  vanishes identically on  $X_0$ , then there exists  $\nu \in \mathbf{Z}_{>0}$  such that  $t^{-\nu} \eta_{\mathcal{X}/\Delta}$  is a nowhere vanishing holomorphic section of  $\mathcal{F}^3$ . Replacing  $\Xi$  by  $f^* t^{-\nu} \cdot \Xi$  in this case, we may and will assume that

$$(3.13) \quad \Xi|_{X_0} \in H^0(X_0, K_{\mathcal{X}|_{X_0}}) \setminus \{0\}.$$

Namely, there is at least one irreducible component of  $X_0$ , on which  $\Xi$  does not vanish. Then  $\eta_{\mathcal{X}/\Delta}$  is a nowhere vanishing holomorphic section of  $\mathcal{F}^3$  and hence

$$(3.14) \quad \log \|\eta_{\mathcal{X}/\Delta}(t)\|_{L^2} = O(\log(-\log|t|)) \quad (t \rightarrow 0)$$

by Proposition 2.2.

**Lemma 3.8.** *The divisor  $\text{div}(\Xi)$  is independent of the choice of  $\Xi \in \Gamma(\mathcal{X}|_{\Delta}, K_{\mathcal{X}/C})$  satisfying (3.12), (3.13).*

*Proof.* Let  $\Xi \otimes (f^* dt)^{-1}$  and  $\Xi' \otimes (f^* dt)^{-1}$  be holomorphic 4-forms on  $\mathcal{X}$  satisfying (3.12), (3.13). Then the ratio  $\Xi/\Xi'$  descends to a nowhere vanishing holomorphic function on  $\Delta^*$ . Since both  $\Xi$  and  $\Xi'$  correspond to nowhere vanishing holomorphic section of the line bundle  $\mathcal{F}^3$ , we conclude that  $\Xi/\Xi'$  is a nowhere vanishing holomorphic function on  $\Delta$ . Hence  $\text{div}(\Xi) = \text{div}(\Xi')$ .  $\square$

After Lemma 3.8, the following definition makes sense.

**Definition 3.9.** The *normalized canonical divisor*  $\mathfrak{K}_{(\mathcal{X}, X_0)}$  of  $(\mathcal{X}, X_0)$  is defined as

$$\mathfrak{K}_{(\mathcal{X}, X_0)} := \text{div}(\Xi), \quad \text{Supp}(\mathfrak{K}_{(\mathcal{X}, X_0)}) \subsetneq \text{Supp}(X_0),$$

where  $\Xi$  satisfies (3.12), (3.13).

To describe the asymptotic behavior of  $A(X_t, g_t)$  as  $t \rightarrow 0$ , we use the notation in Section 3.4. Recall that  $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is a resolution of the indeterminacy of the Gauss map  $\mu: \mathcal{X} \setminus \Sigma_f \rightarrow \mathbf{P}(T\mathcal{X})^\vee$  as in Remark 3.4, that  $\tilde{\mu}: \tilde{\mathcal{X}} \rightarrow \mathbf{P}(T\mathcal{X})^\vee$  is the resolved Gauss map, and that  $U \rightarrow \mathbf{P}(T\mathcal{X})^\vee$  is the universal hyperplane bundle.

We set

$$\tilde{f} := f \circ \sigma$$

and we get a new family  $\tilde{f}: \tilde{\mathcal{X}} \rightarrow C$ , whose central fiber  $\tilde{X}_0 := \tilde{f}^{-1}(0)$  is a possibly *non-reduced* normal crossing divisor. Hence  $\tilde{f}: \tilde{\mathcal{X}} \rightarrow C$  is not necessarily a semi-stable degeneration. Let  $\Sigma_{\tilde{f}}$  be the divisor of  $\tilde{\mathcal{X}}$  defined as the critical locus of  $\tilde{f}$ : If  $\tilde{X}_0 = \sum_{i=1}^k m_i E_i$  with  $E_i$  being an irreducible divisor of  $\tilde{\mathcal{X}}$  and  $m_i \in \mathbf{Z}_{>0}$ , then

$$\Sigma_{\tilde{f}} := \text{div}(d\tilde{f}) = \sum_{i=1}^k (m_i - 1) E_i.$$

Since the resolution  $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is canonically defined,  $\Sigma_{\tilde{f}}$  is determined by the function germ of  $f$  near  $\Sigma_f$ .

**Proposition 3.10.** *The following holds as  $t \rightarrow 0$ :*

$$\log A(X_t, g_t) = -\frac{1}{12} \left( \int_{\sigma^* \mathfrak{K}(\mathcal{X}, X_0) - \Sigma_{\tilde{f}}} \tilde{\mu}^* c_3(U) \right) \log |t|^2 + O(\log(-\log |t|)).$$

*Proof.* Let  $\chi(X_{\text{gen}})$  be the topological Euler number of a general fiber of  $f: \mathcal{X} \rightarrow C$ . Let  $g_U$  be the Hermitian metric on  $U$  induced from the Hermitian metric  $\Pi^* g^{\mathcal{X}}$  on  $\Pi^* T\mathcal{X}$  via the inclusion  $U \subset \Pi^* T\mathcal{X}$  and let  $c_3(U)$  be the top Chern form of  $(U, g_U)$ . Define the function  $A(\mathcal{X}/\Delta)$  on  $\Delta^*$  by  $A(\mathcal{X}/\Delta)(t) := A(X_t, g_t)$ . By [7, Eq. (7.12)], we have

$$\begin{aligned} \log A(\mathcal{X}/\Delta) &= -\frac{1}{12} \tilde{f}_* \left[ \log \sigma^* \left( \frac{\|\Xi\|^2}{\|d\tilde{f}\|^2} \right) \tilde{\mu}^* c_3(U, g_U) \right] + \frac{\chi(X_{\text{gen}})}{12} \log \|\eta_{\mathcal{X}/\Delta}\|_{L^2}^2 \\ &= -\frac{1}{12} \tilde{f}_* \left[ \log \left( \frac{\|\sigma^* \Xi\|^2}{\|d\tilde{f}\|^2} \right) \tilde{\mu}^* c_3(U, g_U) \right] + O(\log(-\log |t|)) \\ &= -\frac{1}{12} \left( \int_{\sigma^* \mathfrak{K}(\mathcal{X}, X_0) - \Sigma_{\tilde{f}}} \tilde{\mu}^* c_3(U, g_U) \right) \log |t|^2 + O(\log(-\log |t|)), \end{aligned}$$

where the second equality follows from (3.14) and the third equality follows from [18, Cor. 4.6] and the equalities of divisors  $\text{div}(\sigma^* \Xi) = \sigma^* \mathfrak{K}(\mathcal{X}, X_0)$ ,  $\Sigma_{\tilde{f}} = \text{div}(d\tilde{f})$  on  $\tilde{\mathcal{X}}$ . This completes the proof.  $\square$

**3.8. Asymptotic behavior of BCOV invariants for semi-stable degenerations.** Define

$$\rho(f, \Sigma_f) := -3a_0(f, \Sigma_f) + 2a_1(f, \Sigma_f) - \chi(Rf_* \mathcal{Q}|_{\Delta}) + \frac{1}{12} \int_{\Sigma_{\tilde{f}}} \tilde{\mu}^* c_3(U) \in \mathbf{Q},$$

$$\kappa(f, \Sigma_f, \mathfrak{K}(\mathcal{X}, X_0)) := \int_{\pi^* \mathfrak{K}(\mathcal{X}, X_0)} \tilde{\mu}^* c_3(U) \in \mathbf{Z}.$$

Since there is a canonical way of resolving the indeterminacy of the Gauss map  $\mu$  for the semi-stable degeneration  $f: \mathcal{X}|_{\Delta} \rightarrow \Delta$  as explained in Remark 3.4,  $\rho(f, \Sigma_f)$  is determined by the function germ of  $f$  near  $\Sigma_f$ , whereas  $\kappa(f, \Sigma_f, \mathfrak{K}(\mathcal{X}, X_0))$  is determined by the function germ of  $f$  near  $\Sigma_f$  and the normalized canonical divisor  $\mathfrak{K}(\mathcal{X}, X_0)$ .

**Theorem 3.11.** *The following holds as  $t \rightarrow 0$ :*

$$\log \tau_{\text{BCOV}}(X_t) = \left\{ \rho(f, \Sigma_f) - \frac{1}{12} \kappa(f, \Sigma_f, \mathfrak{K}(\mathcal{X}, X_0)) \right\} \log |t|^2 + O(\log(-\log |t|)).$$

*Proof.* Since the Kähler metric  $g_t$  is induced from the Kähler metric  $g^{\mathcal{X}}$  on  $\mathcal{X}$ , the functions on  $\Delta^*$

$$t \mapsto \text{Vol}(X_t, g_t), \quad t \mapsto \text{Vol}_{L^2}(H^2(X_t, \mathbf{Z}), [c_1(\mathcal{L}_t)])$$

are constant by [7, Lemma 4.12]. Hence there is a constant  $C > 0$  such that

$$(3.15) \quad \log \tau_{\text{BCOV}}(X_t) = \log T_{\text{BCOV}}(X_t, g_t) + \log A(X_t, g_t) + C$$

for all  $t \in \Delta^*$ . Substituting the formulae in Theorem 3.7 and Proposition 3.10 into (3.15), we get the result.  $\square$

### 3.9. Asymptotic behavior of BCOV invariants for general degenerations.

By Theorem 3.11, we get the rationality of the coefficient of the logarithmic divergence of  $\log \tau_{\text{BCOV}}$  for general one-parameter degenerations. In this subsection, we do not assume that  $f: \mathcal{X}|_{\Delta} \rightarrow \Delta$  is a semi-stable degeneration.

**Theorem 3.12.** *Let  $f: \mathcal{X} \rightarrow C$  be a surjective morphism from an irreducible projective fourfold  $\mathcal{X}$  to a compact Riemann surface  $C$ . If there is a finite subset  $\Delta_f \subset C$  such that  $f|_{C \setminus \Delta_f}: \mathcal{X}|_{C \setminus \Delta_f} \rightarrow C \setminus \Delta_f$  is a smooth morphism and such that  $X_t := f^{-1}(t)$  is a Calabi-Yau threefold for all  $t \in C \setminus \Delta_f$ , then for every  $0 \in \Delta_f$ , there exists a rational number  $\alpha \in \mathbf{Q}$  such that*

$$\log \tau_{\text{BCOV}}(X_t) = \alpha \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0),$$

where  $t$  is a local parameter of  $C$  centered at 0. Let  $g: (\mathcal{Y}, Y_0) \rightarrow (B, 0)$  be a semi-stable reduction of  $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$ :

$$\begin{array}{ccc} (\mathcal{Y}, Y_0) & \xrightarrow{\Phi} & (\mathcal{X}, X_0) \\ g \downarrow & & \downarrow f \\ (B, 0) & \xrightarrow{\phi} & (C, 0). \end{array}$$

Then  $\alpha$  is given by

$$\alpha = \frac{1}{\deg\{\phi: (B, 0) \rightarrow (C, 0)\}} \left\{ \rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) \right\}.$$

*Proof.* By the definition of semi-stable reduction [13, Chap. II],  $\mathcal{Y}$  is a smooth projective fourfold and  $B$  is a compact Riemann surface such that  $\mathcal{Y}|_{B^*} \cong \mathcal{X}|_{C^*} \times_{C^*} B^*$  and the divisor  $Y_0 = g^{-1}(0)$  is reduced and normal crossing. Here we set  $B^* := B \setminus \{0\}$  and  $C^* := C \setminus \{0\}$ . By choosing an appropriate local parameter  $s$  of  $(B, 0)$ , we may assume that  $\phi(s) = s^\nu$ . Since  $Y_s \cong X_{\phi(s)} = X_{s^\nu}$ , the result follows from Theorem 3.11 applied to the semi-stable degeneration of Calabi-Yau threefolds  $g: (\mathcal{Y}, Y_0) \rightarrow (B, 0)$ .  $\square$

In [7, Th. 9.1], a weaker version of Theorem 3.12 was proved, where  $\alpha$  was implicit and real.

## 4. A LOCALITY OF THE LOGARITHMIC SINGULARITY

In this section, we prove a certain locality of the coefficient  $\alpha$  in Theorem 3.12.

**Set up** Let  $\mathcal{X}$  and  $\mathcal{X}'$  be normal irreducible projective fourfolds. Let  $C$  and  $C'$  be compact Riemann surfaces. Let  $f: \mathcal{X} \rightarrow C$  and  $f': \mathcal{X}' \rightarrow C'$  be surjective holomorphic maps. Let  $\overline{\Sigma}_f|_{\mathcal{X} \setminus \text{Sing } \mathcal{X}}$  (resp.  $\overline{\Sigma}_{f'}|_{\mathcal{X}' \setminus \text{Sing } \mathcal{X}'}$ ) be the closure of the critical locus of  $f|_{\mathcal{X} \setminus \text{Sing } \mathcal{X}}$  (resp.  $f'|_{\mathcal{X}' \setminus \text{Sing } \mathcal{X}'}$ ) in  $\mathcal{X}$  (resp.  $\mathcal{X}'$ ). Define the critical loci of  $f$  and  $f'$  as

$$\Sigma_f := \text{Sing } \mathcal{X} \cup \overline{\Sigma}_f|_{\mathcal{X} \setminus \text{Sing } \mathcal{X}}, \quad \Sigma_{f'} := \text{Sing } \mathcal{X}' \cup \overline{\Sigma}_{f'}|_{\mathcal{X}' \setminus \text{Sing } \mathcal{X}'}$$

and the discriminant loci of  $f$  and  $f'$  as

$$\Delta_f := f(\Sigma_f), \quad \Delta_{f'} := f'(\Sigma_{f'}).$$

Let  $0 \in \Delta_f$  and  $0' \in \Delta_{f'}$ . Let  $V$  (resp.  $V'$ ) be a neighborhood of 0 (resp.  $0'$ ) in  $C$  (resp.  $C'$ ) such that  $V \cong \Delta$  and  $V \cap \Delta_f = \{0\}$  (resp.  $V' \cong \Delta$  and  $V' \cap \Delta_{f'} = \{0'\}$ ).

In the rest of this section, we make the following:

**Assumption**

- (A1)  $\Delta_f \neq C$ ,  $\Delta_{f'} \neq C'$ ,  $\dim \Sigma_f \leq 2$ ,  $\dim \Sigma_{f'} \leq 2$ , and  $X_0$  and  $X'_{0'}$  are irreducible.
- (A2)  $X_t$  and  $X'_{t'}$  are Calabi-Yau threefolds for all  $t \in C \setminus \Delta_f$  and  $t' \in C' \setminus \Delta_{f'}$ .
- (A3)  $f^{-1}(V) \setminus \Sigma_f$  carries a nowhere vanishing canonical form  $\Xi$ . Similarly,  $(f')^{-1}(V') \setminus \Sigma_{f'}$  carries a nowhere vanishing canonical form  $\Xi'$ .
- (A4) The function germ of  $f$  near  $\Sigma_f \cap f^{-1}(V)$  and the function germ of  $f'$  near  $\Sigma_{f'} \cap (f')^{-1}(V')$  are isomorphic. Namely, there exist a neighborhood  $O$  of  $\Sigma_f \cap f^{-1}(V)$  in  $f^{-1}(V)$ , a neighborhood  $O'$  of  $\Sigma_{f'} \cap (f')^{-1}(V')$  in  $(f')^{-1}(V')$ , and an isomorphism  $\varphi: O \rightarrow O'$  such that  $f|_O = f' \circ \varphi|_{O'}$ .

By (A3), (A4), the ratio  $\varphi^*(\Xi'|_{O'})/(\Xi|_O)$  is a nowhere vanishing holomorphic function on  $O \setminus \Sigma_f$ . By (A1) and the normality of  $\mathcal{X}$ , the ratio  $\varphi^*(\Xi'|_{O'})/(\Xi|_O)$  extends to a nowhere vanishing holomorphic function on  $O$ . Hence we have the following equality of divisors on  $O$

$$(4.1) \quad \operatorname{div}(\Xi) = \varphi^* \operatorname{div}(\Xi').$$

For  $z \in C$  and  $z' \in C'$ , we set  $X_z := f^{-1}(z)$  and  $X'_{z'} := (f')^{-1}(z')$ . For  $z \in C \setminus \Delta_f$  and  $z' \in C' \setminus \Delta_{f'}$ , the BCOV invariants  $\tau_{\operatorname{BCOV}}(X_z)$  and  $\tau_{\operatorname{BCOV}}(X'_{z'})$  are well defined. Let  $0 \in \Delta_f$  and  $0' \in \Delta_{f'}$ . A local parameter of  $C$  (resp.  $C'$ ) centered at  $0$  (resp.  $0'$ ) is denoted by  $t$ . Hence  $t$  is a generator of the maximal ideal of  $\mathcal{O}_{C,0}$  and  $\mathcal{O}_{C',0'}$ . By Theorem 3.12, the functions  $t \mapsto \log \tau_{\operatorname{BCOV}}(X_t)$  and  $t \mapsto \log \tau_{\operatorname{BCOV}}(X'_t)$  have logarithmic singularities at  $0$  and  $0'$ , respectively.

**Theorem 4.1.** *Under (A1)–(A4),  $\log \tau_{\operatorname{BCOV}}(X_t)$  and  $\log \tau_{\operatorname{BCOV}}(X'_t)$  have the same logarithmic singularities at  $t = 0$ :*

$$\lim_{t \rightarrow 0} \frac{\log \tau_{\operatorname{BCOV}}(X_t)}{\log |t|} = \lim_{t \rightarrow 0} \frac{\log \tau_{\operatorname{BCOV}}(X'_t)}{\log |t|}.$$

In particular,

$$\log \tau_{\operatorname{BCOV}}(X_t) - \log \tau_{\operatorname{BCOV}}(X'_t) = O(\log(-\log |t|)) \quad (t \rightarrow 0).$$

*Proof. (Step 1)* By Hironaka, there exists a succession of blowing-ups  $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  inducing an isomorphism between  $\tilde{\mathcal{X}} \setminus \sigma^{-1}(\Sigma_f)$  and  $\mathcal{X} \setminus \Sigma_f$  such that  $\tilde{X}_0 := (f \circ \sigma)^{-1}(0)$  is a normal crossing divisor of  $\tilde{\mathcal{X}}$ . Let  $\tilde{X}_0 = D_0 \cup D_1 \cup \cdots \cup D_l$  be the irreducible decomposition. We may and will assume that all  $D_\alpha$ 's are smooth hypersurfaces of  $\tilde{\mathcal{X}}$  and that  $D_0$  is the proper transform of  $X_0$ . Then  $D_1 \cup \cdots \cup D_l = \sigma^{-1}(\Sigma_f) \subset \sigma^{-1}(O)$ , and  $\sigma$  induces an isomorphism from  $D_0 \setminus \bigcup_{\alpha > 0} D_\alpha$  to  $X_0 \setminus \Sigma_f$ .

Identify the pairs  $(O, \Sigma_f)$  and  $(O', \Sigma_{f'})$  via  $\varphi$ . We set

$$\tilde{\mathcal{X}}' := (\mathcal{X}' \setminus \Sigma_{f'}) \cup \sigma^{-1}(O),$$

where  $\sigma^{-1}(O \setminus \Sigma_f)$  and  $O' \setminus \Sigma_{f'}$  are identified by the isomorphism  $\varphi \circ \sigma$ . Then  $\tilde{\mathcal{X}}'$  is a smooth fourfold equipped with the projection  $\sigma': \tilde{\mathcal{X}}' \rightarrow \mathcal{X}'$  defined by  $\sigma' := \operatorname{id}$  on  $\mathcal{X}' \setminus \Sigma_{f'}$  and by  $\varphi \circ \sigma$  on  $\sigma^{-1}(O)$ . Since  $\sigma: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is a succession of blowing-ups, so is  $\sigma': \tilde{\mathcal{X}}' \rightarrow \mathcal{X}'$ . We define  $\tilde{f} := f \circ \sigma$  and  $\tilde{f}' := f' \circ \sigma'$ , whose critical loci are denoted by  $\Sigma_{\tilde{f}}$  and  $\Sigma_{\tilde{f}'}$ , respectively. Then  $\Sigma_{\tilde{f}} \subset \sigma^{-1}(O)$  and  $\Sigma_{\tilde{f}'} \subset (\sigma')^{-1}(O')$ .

We set  $\tilde{\varphi} := \operatorname{id}_{\sigma^{-1}(O)}$ . Then  $\varphi$  lifts to an isomorphism  $\tilde{\varphi}: \sigma^{-1}(O) \cong (\sigma')^{-1}(O')$  such that  $\tilde{f}' \circ \tilde{\varphi} = \tilde{f}$  on  $\sigma^{-1}(O)$ . Let  $D'_0 \subset \tilde{\mathcal{X}}'$  be the proper transform of  $X'_0 \subset \mathcal{X}'$ .

Since  $\tilde{f}^{-1}(0) = D_0 \cup D_1 \cup \dots \cup D_l$  and  $\sigma^{-1}(\Sigma_f) = D_1 \cup \dots \cup D_l$ , we have the irreducible decomposition  $(\tilde{f}')^{-1}(0') = D'_0 \cup D'_1 \cup \dots \cup D'_l$  with  $(\sigma')^{-1}(\Sigma_{f'}) = D'_1 \cup \dots \cup D'_l \subset (\sigma')^{-1}(O)$ , where we set  $D'_\alpha := \tilde{\varphi}(D_\alpha)$ . By (4.1) and the equality  $\sigma' \circ \tilde{\varphi} = \varphi \circ \sigma$ , we get

$$(4.2) \quad \operatorname{div}(\sigma^* \Xi) = \tilde{\varphi}^* \operatorname{div}((\sigma')^* \Xi') \subset \sigma^{-1}(O).$$

(Step 2) Let  $d \in \mathbf{Z}_{>0}$ . Let  $\pi_d: (C_d, 0_d) \rightarrow (C, 0)$  (resp.  $\pi'_d: (C'_d, 0'_d) \rightarrow (C', 0')$ ) be a ramified covering with ramification index  $d$  at  $0_d \in C_d$  (resp.  $0'_d \in C'_d$ ). Let  $\tilde{\mathcal{X}}_d$  (resp.  $\tilde{\mathcal{X}}'_d$ ) be the normalization of the fibered product  $\tilde{\mathcal{X}} \times_C C_d$  (resp.  $\tilde{\mathcal{X}}' \times_{C'} C'_d$ ) and set  $\tilde{f}_d := \operatorname{pr}_2: \tilde{\mathcal{X}}_d \rightarrow C_d$  (resp.  $\tilde{f}'_d := \operatorname{pr}_2: \tilde{\mathcal{X}}'_d \rightarrow C'_d$ ). Let  $\tilde{O}_d$  (resp.  $\tilde{O}'_d$ ) be the open subset of  $\tilde{\mathcal{X}}_d$  (resp.  $\tilde{\mathcal{X}}'_d$ ) defined as  $\operatorname{pr}_1^{-1}(\sigma^{-1}(O))$  (resp.  $(\operatorname{pr}'_1)^{-1}((\sigma')^{-1}(O'))$ ). Then  $\tilde{\varphi}: \sigma^{-1}(O) \cong (\sigma')^{-1}(O')$  lifts to an isomorphism  $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$  such that  $\tilde{f}_d = \tilde{f}'_d \circ \tilde{\varphi}_d$ .

Define  $U_d := \tilde{\mathcal{X}}_d \setminus \tilde{f}_d^{-1}(0_d)$  and  $U'_d := \tilde{\mathcal{X}}'_d \setminus (\tilde{f}'_d)^{-1}(0'_d)$ . By [13, Chap. II, §3], the pairs  $(\tilde{\mathcal{X}}_d, U_d)$  and  $(\tilde{\mathcal{X}}'_d, U'_d)$  are toroidal embeddings. (See [13, Chap. II §1] for the notion of toroidal embeddings.) Let  $\tilde{f}_d^{-1}(0_d) = E_0 \cup E_1 \cup \dots \cup E_m$  (resp.  $(\tilde{f}'_d)^{-1}(0'_d) = E'_0 \cup E'_1 \cup \dots \cup E'_m$ ) be the irreducible decomposition, where  $E_0$  (resp.  $E'_0$ ) is the component corresponding to the proper transforms of  $X_0$  (resp.  $X'_0$ ) in  $\tilde{\mathcal{X}}$  (resp.  $\tilde{\mathcal{X}}'$ ). Then  $m \geq n$ . Since  $X_0 \setminus \Sigma_f \cong D_0 \setminus \bigcup_{\alpha>0} D_\alpha$  is reduced and smooth, we have  $E_0 \setminus \bigcup_{\beta>0} E_\beta \cong D_0 \setminus \bigcup_{\alpha>0} D_\alpha \cong X_0 \setminus \Sigma_f$ . When  $\beta > 0$ , we deduce from [13, Chap. II §3] that  $\operatorname{pr}_1(E_\beta) = D_{\alpha(\beta)}$  for some  $\alpha(\beta) > 0$ . Similarly,  $E'_0 \setminus \bigcup_{\beta>0} E'_\beta \cong X'_0 \setminus \Sigma_{f'}$  and  $\operatorname{pr}_1(E'_\beta) = D'_{\alpha(\beta)}$  ( $\alpha(\beta) > 0$ ) for  $\beta > 0$ .

(Step 3) By an appropriate choice of  $d \in \mathbf{Z}_{>0}$ , there exists a sheaf of ideals  $\mathcal{I}_d \subset \mathcal{O}_{\tilde{\mathcal{X}}_d}$  with  $\mathcal{I}_d|_{U_d} = \mathcal{O}_{U_d}$ , whose blowing-up  $\varpi: \mathcal{Y}_d \rightarrow \tilde{\mathcal{X}}_d$  provides a semi-stable reduction of  $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$ , i.e., the following commutative diagram (cf. [13, Chap. II])

$$\begin{array}{ccccccc} \mathcal{Y}_d & \xrightarrow{\varpi} & \tilde{\mathcal{X}}_d & \xrightarrow{\operatorname{pr}_1} & \tilde{\mathcal{X}} & \xrightarrow{\sigma} & \mathcal{X} \\ g_d \downarrow & & \tilde{f}_d \downarrow & & \tilde{f} \downarrow & & f \downarrow \\ C_d & \xrightarrow{\operatorname{id}} & C_d & \xrightarrow{\pi_d} & C & \xrightarrow{\operatorname{id}} & C, \end{array}$$

where  $\mathcal{Y}_d$  is smooth and  $g_d^{-1}(0_d)$  is a *reduced*, normal crossing divisor of  $\mathcal{Y}_d$ . We define  $E_\beta^o := E_\beta \setminus \bigcup_{\beta' \neq \beta} E_{\beta'}$ . Since the ideal sheaf  $\mathcal{I}_d$  is of the form as in [13, p. 91 last line], we deduce from [13, Th. 9\*] that there exists  $\nu_0 \in \mathbf{Z}$  with

$$(4.3) \quad \mathcal{I}_d|_{U_d \cup E_0^o} \cong \mathcal{O}_{U_d \cup E_0^o}(\nu_0 E_0^o).$$

Since  $E_0^o$  is a smooth divisor of  $\tilde{\mathcal{X}}_d \setminus \bigcup_{\beta>0} E_\beta = U_d \cup E_0^o$ , we deduce from (4.3) and the definition of blowing-up of sheaf of ideals that the maps  $\varpi: \varpi^{-1}(E_0^o) \rightarrow E_0^o$  and  $\varpi: \mathcal{Y}_d \setminus \varpi^{-1}(E_1 \cup \dots \cup E_m) \rightarrow \tilde{\mathcal{X}}_d \setminus (E_1 \cup \dots \cup E_m)$  are isomorphisms. Write  $g_d^{-1}(0_d) = F_0 + \dots + F_n$ , where every  $F_i$  is irreducible and  $F_0$  is the proper transform of  $E_0$ . Then  $n \geq m$  and  $F_1 \cup \dots \cup F_n = \varpi^{-1}(E_1 \cup \dots \cup E_m) \subset \varpi^{-1}(\tilde{O}_d)$ .

(Step 4) We define the sheaf of ideals  $\mathcal{I}'_d \subset \mathcal{O}_{\tilde{\mathcal{X}}'_d}$  by

$$\mathcal{I}'_d|_{\tilde{\mathcal{X}}'_d \setminus \bigcup_{\beta>0} E'_\beta} = \mathcal{O}_{\tilde{\mathcal{X}}'_d \setminus \bigcup_{\beta>0} E'_\beta}(\nu_0 E'_0), \quad \mathcal{I}'_d|_{O'_d} = (\varphi_d)_* \mathcal{I}_d.$$

Then  $\mathcal{I}'_d|_{U'_d} = \mathcal{O}_{U'_d}$ . Let  $\varpi': \mathcal{Y}' \rightarrow \tilde{\mathcal{X}}'_d$  be the blowing-up of  $\mathcal{I}'_d$  and set  $g'_d := \tilde{f}'_d \circ \varpi'$ . Since the map  $\varpi': (\varpi')^{-1}(\tilde{O}'_d) \rightarrow \tilde{O}'_d$  is identified with  $\varpi: \varpi^{-1}(\tilde{O}_d) \rightarrow \tilde{O}_d$  via the

identification  $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$ , we get the isomorphism of divisors

$$(g'_d)^{-1}(0'_d) \cap (\varpi')^{-1}(\tilde{O}'_d) \cong (F_0 \cap \tilde{O}_d) + F_1 + \cdots + F_n,$$

which is a reduced normal crossing divisor of  $\varpi^{-1}(\tilde{O}_d)$ . Let  $F'_0 \subset (g'_d)^{-1}(0_d)$  be the proper transform of  $E'_0 \subset \tilde{\mathcal{X}}'_d$  and let  $F'_\gamma \subset (g'_d)^{-1}(0_d)$  be the irreducible component corresponding to  $F_\gamma$  for  $\gamma > 0$ . Then  $F'_0 \cap (\varpi')^{-1}(\tilde{O}'_d) \cong F_0 \cap \tilde{O}_d$ . Since the map  $\varpi': \mathcal{Y}' \setminus (\varpi')^{-1}(\tilde{O}'_d) \rightarrow \tilde{\mathcal{X}}'_d \setminus \tilde{O}'_d$  is an isomorphism,  $F'_0 \setminus (\varpi')^{-1}(\tilde{O}'_d) \cong X'_0 \setminus \tilde{O}'$  is a smooth divisor of  $\mathcal{Y}' \setminus (\varpi')^{-1}(\tilde{O}'_d)$ . Hence  $(g'_d)^{-1}(0_d) = F'_0 + \cdots + F'_n$  is a reduced normal crossing divisor of  $\mathcal{Y}'$ . Thus  $g'_d: \mathcal{Y}'_d \rightarrow C_d$  is a semi-stable reduction of  $f': \mathcal{X}' \rightarrow C'$ . We have the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{Y}'_d & \xrightarrow{\varpi'} & \tilde{\mathcal{X}}'_d & \xrightarrow{\text{pr}_1} & \tilde{\mathcal{X}}' & \xrightarrow{\sigma'} & \mathcal{X}' \\ g'_d \downarrow & & \tilde{f}'_d \downarrow & & \tilde{f}' \downarrow & & f' \downarrow \\ C'_d & \xrightarrow{\text{id}} & C'_d & \xrightarrow{\pi_d} & C' & \xrightarrow{\text{id}} & C'. \end{array}$$

(Step 5) Set  $O_d := \varpi^{-1}(\tilde{O}_d)$  and  $O'_d := (\varpi')^{-1}(\tilde{O}'_d)$ . Let  $\varphi_d: O_d \cong O'_d$  be the isomorphism induced by  $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$ . Since

$$\Sigma_{g_d} \subset O_d, \quad \Sigma_{g'_d} = \varphi_d(\Sigma_{g_d}) \subset O'_d, \quad (g_d, O_d) = (g'_d \circ \varphi_d, O_d)$$

by construction, we get the following equality by the definition in Section 3.8

$$(4.4) \quad \rho(g_d, \Sigma_{g_d}) = \rho(g'_d, \Sigma_{g'_d}).$$

(Step 6) Set  $\psi := \sigma \circ \text{pr}_1 \circ \varpi: \mathcal{Y}_d \rightarrow \mathcal{X}$  and  $\psi' := \sigma' \circ \text{pr}_1 \circ \varpi': \mathcal{Y}'_d \rightarrow \mathcal{X}'$ . Let  $\Upsilon$  (resp.  $\Upsilon'$ ) be a canonical form defined near  $g_d^{-1}(0_d)$  (resp.  $(g'_d)^{-1}(0'_d)$ ) and satisfying (3.12), (3.13). Then there exist  $a_\gamma, a'_\gamma \in \mathbf{Z}_{\geq 0}$  for  $0 \leq \gamma \leq n$  such that

$$(4.5) \quad \text{div}(\Upsilon) = \sum_{\gamma=0}^n a_\gamma F_\gamma, \quad \text{div}(\Upsilon') = \sum_{\gamma=0}^n a'_\gamma F'_\gamma.$$

Since  $\psi^* \Xi$  (resp.  $(\psi')^* \Xi'$ ) is a (possibly meromorphic) 4-form defined on a neighborhood of  $g_d^{-1}(0_d)$  (resp.  $(g'_d)^{-1}(0'_d)$ ), whose possible zeros and poles are supported on  $g_d^{-1}(0_d)$  (resp.  $(g'_d)^{-1}(0'_d)$ ), we can express

$$(4.6) \quad \text{div}(\psi^* \Xi) = \sum_{\gamma=0}^n b_\gamma F_\gamma, \quad \text{div}((\psi')^* \Xi') = \sum_{\gamma=0}^n b'_\gamma F'_\gamma,$$

where  $b_\gamma, b'_\gamma \in \mathbf{Z}$  for  $0 \leq \gamma \leq n$ . Since  $\Xi$  (resp.  $\Xi'$ ) is nowhere vanishing on  $f^{-1}(V) \setminus O$  (resp.  $(f')^{-1}(V') \setminus O'$ ) by assumption and since  $\psi$  (resp.  $\psi'$ ) has ramification index  $d$  along  $F_0 \setminus \bigcup_{\gamma>0} F_\gamma$  (resp.  $F'_0 \setminus \bigcup_{\gamma>0} F'_\gamma$ ),  $\psi^* \Xi$  (resp.  $(\psi')^* \Xi'$ ) has zeros of order  $d-1$  on the proper transform of  $E_0$  (resp.  $E'_0$ ). Hence

$$(4.7) \quad b_0 = b'_0 = d-1.$$

Since the map  $\varpi': O'_d \rightarrow \tilde{O}'_d$  is identified with  $\varpi: O_d \rightarrow \tilde{O}_d$  via the identification  $\tilde{\varphi}_d: \tilde{O}_d \cong \tilde{O}'_d$ , we get by (4.2)

$$\text{div}(\psi^* \Xi) \cap O_d = \varphi_d^*(\text{div}((\psi')^* \Xi')) \cap O_d.$$

Hence

$$(4.8) \quad b_\gamma = b'_\gamma \quad (\gamma > 0).$$

Since  $\varphi^*(\Xi' \wedge \overline{\Xi}')/(\Xi \wedge \overline{\Xi})$  is a nowhere vanishing positive function on  $O$  by (4.1), there exist constants  $C_1, C_2 > 0$  such that for all  $t$  with  $0 < |t| \ll 1$ ,

$$(4.9) \quad C_1 \int_{X_t \cap O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}} \leq \int_{X'_t \cap O'} \sqrt{-1} \frac{\Xi' \wedge \overline{\Xi}'}{df' \wedge d\overline{f}'} \leq C_2 \int_{X_t \cap O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}}.$$

Since  $\sqrt{-1}\Xi \wedge \overline{\Xi}$  (resp.  $\sqrt{-1}\Xi' \wedge \overline{\Xi}'$ ) is a volume form on  $f^{-1}(V) \setminus O$  (resp.  $(f')^{-1}(V') \setminus O'$ ) and since  $f$  (resp.  $f'$ ) has no critical points on  $f^{-1}(V) \setminus O$  (resp.  $(f')^{-1}(V') \setminus O'$ ), there exist constants  $C_3, C_4 > 0$  such that for all  $t$  with  $0 < |t| \ll 1$ ,

$$(4.10) \quad C_3 \int_{X_t \setminus O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}} \leq \int_{X'_t \setminus O'} \sqrt{-1} \frac{\Xi' \wedge \overline{\Xi}'}{df' \wedge d\overline{f}'} \leq C_4 \int_{X_t \setminus O} \sqrt{-1} \frac{\Xi \wedge \overline{\Xi}}{df \wedge d\overline{f}}.$$

By (4.9), (4.10), there exist constants  $C_5, C_6 > 0$  such that for all  $t$  with  $0 < |t| \ll 1$ ,

$$(4.11) \quad C_5 \left\| \frac{\Xi}{df} \Big|_{X_t} \right\|_{L^2}^2 \leq \left\| \frac{\Xi'}{df'} \Big|_{X'_t} \right\|_{L^2}^2 \leq C_6 \left\| \frac{\Xi}{df} \Big|_{X_t} \right\|_{L^2}^2.$$

(Step 7) Let  $s$  be a local parameter of  $C_d$  and  $C'_d$  centered at  $0_d$  and  $0'_d$ . Since  $\pi_d: C_d \rightarrow C$  and  $\pi'_d: C'_d \rightarrow C'$  has ramification index  $d$  at  $s = 0$ , we may assume

$$(4.12) \quad \pi_d(s) = s^d.$$

By the definition of  $\Upsilon$  (resp.  $\Upsilon'$ ), the map  $s \mapsto (\Upsilon/dg_d)|_{Y_s} \in H^0(Y_s, K_{Y_s})$  (resp.  $s \mapsto (\Upsilon'/dg'_d)|_{Y'_s} \in H^0(Y'_s, K_{Y'_s})$ ) is a nowhere vanishing holomorphic section of  $(g_d)_*K_{Y_d/C_d}$  (resp.  $(g'_d)_*K_{Y'_d/C'_d}$ ) near  $0_d$  (resp.  $0'_d$ ). By (3.14), we get

$$(4.13) \quad \log \left\| \frac{\Upsilon}{dg_d} \Big|_{Y_s} \right\|_{L^2}^2 = O(\log(-\log|s|)), \quad \log \left\| \frac{\Upsilon'}{dg'_d} \Big|_{Y'_s} \right\|_{L^2}^2 = O(\log(-\log|s|))$$

as  $s \rightarrow 0$ , where we set  $Y_s := g_d^{-1}(s)$  and  $Y'_s := (g'_d)^{-1}(s)$ .

Since the fibers  $Y_s$  and  $Y'_s$  are Calabi-Yau threefolds for  $s \neq 0$ , the map  $s \mapsto \psi^*\Xi/dg_d|_{Y_s}$  (resp.  $s \mapsto (\psi')^*\Xi'/dg'_d|_{Y'_s}$ ) is a holomorphic section of  $(g_d)_*K_{Y_d/C_d}$  (resp.  $(g'_d)_*K_{Y'_d/C'_d}$ ) near  $0_d$  (resp.  $0'_d$ ). Hence there exist  $c, c' \in \mathbf{Z}$  and  $\epsilon(s), \epsilon'(s) \in \mathcal{O}(\Delta)$  such that

$$(4.14) \quad \frac{\psi^*\Xi}{\Upsilon} \Big|_{Y_s} = s^c \epsilon(s), \quad \frac{(\psi')^*\Xi'}{\Upsilon'} \Big|_{Y'_s} = s^{c'} \epsilon'(s), \quad \epsilon(0) \neq 0, \quad \epsilon'(0) \neq 0.$$

By (4.5), (4.6), (4.7), (4.8), (4.14), we get

$$(4.15) \quad a_\gamma = b_\gamma + c, \quad a'_\gamma = b'_\gamma + c' = b_\gamma + c'.$$

Since  $Y_s = X_{\pi_d(s)} = X_{s^d}$  and  $Y'_s = X'_{s^d}$  for  $s \neq 0$  by (4.12), we get by (4.13), (4.14), (4.16)

$$\begin{aligned}
\log \left\| \frac{\Xi}{df} \Big|_{X_{s^d}} \right\|_{L^2}^2 &= \log \left\| \psi^* \left\{ \frac{\Xi}{df} \Big|_{X_{s^d}} \right\} \right\|_{L^2}^2 = \log \left\| \frac{\Xi}{d(g_d^d)} \Big|_{Y_s} \right\|_{L^2}^2 \\
&= \log \left( |s|^{-2(d-1)} \left\| \frac{\Xi}{dg_d} \Big|_{Y_s} \right\|_{L^2}^2 \right) + O(1) \\
&= -(d-1) \log |s|^2 + \log \left\| \frac{\psi^* \Xi}{\Upsilon} \Big|_{Y_s} \right\|_{L^2}^2 + \log \left\| \frac{\Upsilon}{dg_d} \Big|_{Y_s} \right\|_{L^2}^2 + O(1) \\
&= (c-d+1) \log |s|^2 + O(\log(-\log |s|)) \quad (s \rightarrow 0).
\end{aligned}$$

Similarly, we get

$$(4.17) \quad \log \left\| \frac{\Xi'}{df'} \Big|_{X'_{s^d}} \right\|_{L^2}^2 = (c'-d+1) \log |s|^2 + O(\log(-\log |s|)) \quad (s \rightarrow 0).$$

Comparing (4.11) and (4.16), (4.17), we get

$$(4.18) \quad c = c'.$$

By (4.15), (4.18), we get  $a_\gamma = a'_\gamma$  for all  $0 \leq \gamma \leq n$ . Hence

$$(4.19) \quad \operatorname{div}(\Upsilon) = \sum_{\gamma=0}^n a_\gamma F_\gamma, \quad \operatorname{div}(\Upsilon') = \sum_{\gamma=0}^n a_\gamma F'_\gamma, \quad a_\gamma \in \mathbf{Z}_{\geq 0}, \quad (\forall \gamma \geq 0).$$

Since  $F_0 \cap O_d$  (resp.  $F_\gamma$  ( $\gamma > 0$ )) is identified with  $F'_0 \cap O'_d$  (resp.  $F'_\gamma$  ( $\gamma > 0$ )) via  $\varphi_d$ , we get by (4.19) and the definition of normalized canonical divisor in Section 3.8 the following equality of divisors via the identification  $\varphi_d: O_d \cong O'_d$ :

$$(4.20) \quad \mathfrak{K}_{(\mathcal{Y}_d, Y_0)} \cap O_d = \varphi_d^*(\mathfrak{K}_{(\mathcal{Y}'_d, Y'_0)} \cap O'_d).$$

Since  $(g_d, \Sigma_{g_d}) = (g'_d \circ \varphi_d, \Sigma_{g_d})$  and  $\Sigma_d = \varphi_d(\Sigma_{g'_d})$  via  $\varphi_d: O_d \cong O'_d$ , we deduce from (4.20) and the definition in Section 3.8 the equality

$$(4.21) \quad \kappa(g_d, \Sigma_{g_d}, \mathfrak{K}_{(\mathcal{Y}_d, Y_0)}) = \kappa(g'_d, \Sigma_{g'_d}, \mathfrak{K}_{(\mathcal{Y}'_d, Y'_0)}).$$

By Theorem 3.11 and (4.4), (4.21), we get

$$(4.22) \quad \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(Y_s)}{\log |s|^2} = \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(Y'_s)}{\log |s|^2}.$$

Since  $Y_s = X_{s^d}$  and  $Y'_s = X'_{s^d}$  for  $s \neq 0$ , the result follows from (4.22).  $\square$

## 5. DEGENERATIONS TO CALABI-YAU VARIETIES WITH ORDINARY DOUBLE POINTS

In this section, we determine the asymptotic behavior of BCOV invariants for the simplest degenerations of Calabi-Yau threefolds, i.e., degenerations to Calabi-Yau varieties with at most ordinary double points (cf. [7, §2]). Recall that an  $n$ -dimensional singularity is an *ordinary double point* if it is isomorphic to the hypersurface singularity at  $0 \in \mathbf{C}^n$  defined by the equation  $z_0^2 + \cdots + z_n^2 = 0$ .

**Definition 5.1.** A complex projective variety  $X$  of dimension 3 is a *Calabi-Yau variety with at most ordinary double points* if the following are satisfied:



- (1) There exists a nowhere vanishing canonical form on  $X \setminus \text{Sing}(X)$ .
- (2)  $X$  is connected and  $H^q(X, \mathcal{O}_X) = 0$  for  $0 < q < 3$ .
- (3)  $\text{Sing}(X)$  consists of at most ordinary double points.

**Theorem 5.2.** *Let  $f: \mathcal{X} \rightarrow C$  be a surjective morphism from a smooth projective fourfold  $\mathcal{X}$  to a compact Riemann surface  $C$ . Let  $\Delta_f \subset C$  be the discriminant locus of  $f: \mathcal{X} \rightarrow C$  and assume that  $X_t := f^{-1}(t)$  is a Calabi-Yau threefold for all  $t \in C \setminus \Delta_f$ . Let  $0 \in \Delta_f$  and let  $t$  be a local parameter of  $C$  centered at 0. If  $X_0$  is a Calabi-Yau variety with at most ordinary double points, then*

$$\log \tau_{\text{BCOV}}(X_t) = \frac{\#\text{Sing } X_0}{6} \log |t|^2 + O(\log(-\log |t|)) \quad (t \rightarrow 0).$$

*Proof.* (Step 1) Since the deformation germ  $f: (\mathcal{X}, X_0) \rightarrow (C, 0)$  is a smoothing of  $X_0$ , we have  $h^{1,2}(X_0) = h^{2,1}(X_0) \geq 1$ . Since  $t \mapsto h^2(X_t, \Omega_{X_t}^1)$  is a locally constant function on  $C$  by [7, Th. 2.11], we get  $h^{1,2}(X_t) = h^{2,1}(X_t) \geq 1$ . When  $h^{1,2}(X_t) = h^{2,1}(X_t) = 1$  and  $\#\text{Sing } X_0 = 1$ , the result was proved in [7, Th. 8.2]. Since there does exist a family of Calabi-Yau threefolds  $f': \mathcal{X}' \rightarrow C'$  with  $h^{1,2}(X'_t) = 1$  such that  $\text{Sing } X'_0$  consists of a unique ordinary double point, e.g. the family of quintic mirror threefolds (cf. [7, §12]), we get the result by Theorem 4.1 and [7, Th. 8.2] when  $\#\text{Sing } X_0 = 1$ .

(Step 2) Fix a family of Calabi-Yau threefolds over a compact Riemann surface  $\mathfrak{f}: \mathfrak{X} \rightarrow \mathfrak{C}$  such that  $\mathfrak{X}_0$ ,  $0 \in \mathfrak{C}$ , is a Calabi-Yau variety with a unique ordinary double point as its singular set. Fix its semi-stable reduction  $\mathfrak{g}: (\mathfrak{Y}, \mathfrak{Y}_0) \rightarrow (\mathfrak{B}, 0)$ : We have a commutative diagram:

$$\begin{array}{ccc} (\mathfrak{Y}, \mathfrak{Y}_0) & \xrightarrow{\Psi} & (\mathfrak{X}, \mathfrak{X}_0) \\ \mathfrak{g} \downarrow & & \downarrow \mathfrak{f} \\ (\mathfrak{B}, 0) & \xrightarrow{\psi} & (\mathfrak{C}, 0). \end{array}$$

Let  $\nu$  be the ramification index of  $\psi: (\mathfrak{B}, 0) \rightarrow (\mathfrak{C}, 0)$ . Let  $\mathfrak{Y}_0 = \mathfrak{E}_0 + \cdots + \mathfrak{E}_n$  be the irreducible decomposition such that  $\mathfrak{E}_0 \setminus \bigcup_{\alpha > 0} \mathfrak{E}_\alpha \cong \mathfrak{X}_0 \setminus \text{Sing } \mathfrak{X}_0$ . Then  $\Psi$  ramifies along  $\mathfrak{E}_0$  with ramification index  $\nu$ . There exist a neighborhood  $\mathfrak{U}$  of  $\text{Sing } \mathfrak{X}_0$  in  $\mathfrak{X}$  and an open subset  $\mathfrak{V}$  of  $\mathfrak{Y}$  such that  $\mathfrak{V} = \Psi^{-1}(\mathfrak{U})$  and  $\mathfrak{E}_1 \cup \cdots \cup \mathfrak{E}_n \subset \mathfrak{V}$ . Then  $\Psi$  induces an isomorphism between  $\mathfrak{E}_0 \setminus \mathfrak{V}$  and  $\mathfrak{X}_0 \setminus \mathfrak{U}$  and the map  $\Psi$  has ramification index  $\nu$  on  $\mathfrak{E}_0 \setminus \bigcup_{\alpha > 0} \mathfrak{E}_\alpha$ .

(Step 3) By an appropriate choices of local parameters  $t$  of  $(\mathfrak{C}, 0)$  and  $s$  of  $(\mathfrak{B}, 0)$ , we may assume  $t = s^\nu$ . By identifying  $\mathfrak{f}$  with  $t \circ \mathfrak{f}$  and  $\mathfrak{g}$  with  $s \circ \mathfrak{g}$ , we have the following equality of functions defined near  $\mathfrak{Y}_0$ :

$$\Psi^* \mathfrak{f} = \mathfrak{g}^\nu.$$

Since  $\mathfrak{X}_0$  is a Calabi-Yau variety with a unique ordinary double point as its singular set and since  $\mathfrak{X}$  is smooth, there exists a nowhere vanishing holomorphic 4-form  $\Xi$  defined on a neighborhood of  $\mathfrak{X}_0$  in  $\mathfrak{X}$ . Since  $\mathfrak{X}_0$  has only canonical singularities, the function

$$t \mapsto \|(\Xi/d\mathfrak{f})|_{\mathfrak{X}_t}\|_{L^2}^2$$

is continuous around  $0 \in \mathfrak{C}$  and  $\|(\Xi/d\mathfrak{f})|_{\mathfrak{X}_0}\|_{L^2}^2 \neq 0$  by e.g. [20, Th. 7.2]. Since

$$\nu \Psi^*(\Xi/d\mathfrak{f}) = \Psi^* \Xi / (\mathfrak{g}^{\nu-1} d\mathfrak{g}),$$

the fact that the function  $t \mapsto \|(\Xi/df)|_{\mathfrak{X}_t}\|_{L^2}^2$  is  $C^0$  and does not vanish at  $t = 0$  implies that the map  $s \mapsto (\Psi^*\Xi/g^{\nu-1}dg)|_{\mathfrak{Y}_s} \in H^0(\mathfrak{Y}_s, K_{\mathfrak{Y}}|_{\mathfrak{Y}_s})$  is a nowhere vanishing holomorphic section of  $\mathfrak{g}_*K_{\mathfrak{Y}}$  defined near  $s = 0$ . Hence  $(\Psi^*\Xi)/g^{\nu-1}$  is a nowhere vanishing holomorphic 4-form defined near  $\mathfrak{Y}_0$  and

$$0 \leq \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)} = \operatorname{div}(\Psi^*\Xi/g^{\nu-1}) = \operatorname{div}(\Psi^*\Xi) - (\nu - 1)\mathfrak{Y}_0.$$

Write  $\operatorname{div}(\Psi^*\Xi) = \sum_{\alpha=0}^n a_{\alpha} \mathfrak{E}_{\alpha}$ ,  $a_{\alpha} \in \mathbf{Z}_{\geq 0}$ . Since  $\mathfrak{Y}_0 = \operatorname{div}(\mathfrak{g}) = \sum_{\alpha=0}^n \mathfrak{E}_{\alpha}$  by the reducedness of  $\mathfrak{Y}_0$ , we get by the effectivity of  $\operatorname{div}(\Psi^*\Xi) - (\nu - 1)\mathfrak{Y}_0$

$$a_{\alpha} \geq \nu - 1 \quad (\forall \alpha \in \{0, 1, \dots, n\}).$$

On the other hand, since  $\Psi$  has ramification index  $\nu$  on  $\mathfrak{E}_0 \setminus \bigcup_{\alpha>0} \mathfrak{E}_{\alpha}$ , we have  $\alpha_0 = \nu - 1$ . Hence we can express

$$(5.1) \quad \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)} = \operatorname{div}(\Psi^*\Xi/g^{\nu-1}|_{\mathfrak{Y}}) = \sum_{\alpha=1}^n b_{\alpha} \mathfrak{E}_{\alpha} \quad (b_{\alpha} \in \mathbf{Z}_{\geq 0}).$$

(Step 4) Consider the general case. Set  $m := \#\operatorname{Sing} X_0 \geq 1$ . Then  $\operatorname{Sing} X_0 = \{p_1, \dots, p_m\}$ . Since every germ  $f \in \mathcal{O}_{X_0, p_i}$  is isomorphic to the germ  $(z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2$  at  $0 \in \mathbf{C}^4$ , it follows from the construction of semi-stable reduction [13, Chap. II §3] that there exists a semi-stable reduction

$$\begin{array}{ccc} (\mathcal{Y}, Y_0) & \xrightarrow{\Phi} & (\mathcal{X}, X_0) \\ g \downarrow & & \downarrow f \\ (B, 0) & \xrightarrow{\phi} & (C, 0) \end{array}$$

with the following properties:

- (i) The ramification index of  $\phi: (B, 0) \rightarrow (C, 0)$  is given by  $\nu$ .
- (ii) For every  $p_i \in \operatorname{Sing} X_0$ , there is a neighborhood  $U_i$  of  $p_i$  in  $\mathcal{X}$  such that  $(f, U_i) \cong (\mathfrak{f}, \mathfrak{U})$  and  $(g, \Phi^{-1}(U_i)) \cong (\mathfrak{g}, \mathfrak{Y})$ .
- (iii) Set  $V_i := \Phi^{-1}(U_i)$ . Then  $Y_0 \setminus \bigcup_{i=1}^m V_i \cong X_0 \setminus \bigcup_{i=1}^m U_i$ .

By (ii), the irreducible component of  $Y_0 = g^{-1}(0)$  contained in  $V_i$  can be expressed as  $E_1^{(i)} + \dots + E_n^{(i)}$  and satisfy  $E_0 \cap V_i + E_1^{(i)} + \dots + E_n^{(i)} \cong \mathfrak{E}_0 \cap \mathfrak{Y} + \mathfrak{E}_1 + \dots + \mathfrak{E}_n$ , where  $E_{\alpha}^{(i)} \cong \mathfrak{E}_{\alpha}$  for all  $i = 1, \dots, m$  and  $1 \leq \alpha \leq n$ . This implies that

$$(5.2) \quad \rho(g, \Sigma_g) = \sum_{i=1}^m \rho(g|_{V_i}, \Sigma_{g|_{V_i}}) = m \rho(\mathfrak{g}, \Sigma_{\mathfrak{g}}).$$

(Step 5) Let  $t$  be a local parameter of  $(C, 0)$  and let  $s$  be a local parameter of  $(B, 0)$ . As in Step 3, we may assume  $\phi^*t = s^{\nu}$  and hence

$$\Phi^*f = g^{\nu}.$$

Let  $\omega$  be a nowhere vanishing holomorphic 4-form defined near  $X_0$ . Since  $X_0$  has only canonical singularities, the section of the Hodge bundle  $t \mapsto \omega/df|_{X_t} \in H^0(X_t, K_{X_t})$  is holomorphic and nowhere vanishing. By the same reason as in Step 3, the section of Hodge bundle  $s \mapsto \Phi^*\omega/(g^{\nu-1}dg)|_{Y_s} \in H^0(Y_s, K_{Y_s})$  is holomorphic and nowhere vanishing around  $s = 0$  and we get

$$(5.3) \quad \mathfrak{K}_{(\mathcal{Y}, Y_0)} = \operatorname{div}(\Phi^*\omega/g^{\nu-1}), \quad \operatorname{Supp} \mathfrak{K}_{(\mathcal{Y}, Y_0)} \subset \bigcup_{i=1}^m \bigcup_{\alpha=1}^n E_{\alpha}^{(i)}.$$

We may assume that, under the identification  $(f, U_i) \cong (\mathfrak{f}, \mathfrak{U})$  in (ii), the ratio  $\Xi/\omega$  is a nowhere vanishing holomorphic function on  $\mathfrak{U}$ . Then we get by (5.1), (5.3)

$$(5.4) \quad \mathfrak{K}_{(\mathcal{Y}, Y_0)} = \sum_{i=1}^m \sum_{\alpha=1}^n b_{\alpha} E_{\alpha}^{(i)}.$$

By (ii), (5.4) and the definition of  $\kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)})$ , we get

$$(5.5) \quad \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) = m \kappa(\mathfrak{g}, \Sigma_{\mathfrak{g}}, \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)}).$$

Since  $\mathfrak{X}_{s^{\nu}} = \mathfrak{Y}_s$ , we get by Theorem 3.11 and Step 1

$$\rho(\mathfrak{g}, \Sigma_{\mathfrak{g}}) - \frac{1}{12} \kappa(\mathfrak{g}, \Sigma_{\mathfrak{g}}, \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)}) = \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(\mathfrak{Y}_s)}{\log |s|^2} = \lim_{s \rightarrow 0} \frac{\log \tau_{\text{BCOV}}(\mathfrak{X}_{s^{\nu}})}{\log |s|^2} = \frac{\nu}{12}.$$

Thus we get by (5.2), (5.5)

$$\rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) = m \left\{ \rho(\mathfrak{g}, \Sigma_{\mathfrak{g}}) - \frac{1}{12} \kappa(\mathfrak{g}, \Sigma_{\mathfrak{g}}, \mathfrak{K}_{(\mathfrak{Y}, \mathfrak{Y}_0)}) \right\} = \frac{m\nu}{12}.$$

By Theorem 3.11 again and the relation  $Y_s = X_{s^{\nu}}$ , we get

$$\begin{aligned} \log \tau_{\text{BCOV}}(X_{s^d}) &= \log \tau_{\text{BCOV}}(Y_s) \\ &= \left\{ \rho(g, \Sigma_g) - \frac{1}{12} \kappa(g, \Sigma_g, \mathfrak{K}_{(\mathcal{Y}, Y_0)}) \right\} \log |s|^2 + O(\log(-\log |s|)) \\ &= \frac{m\nu}{12} \log |s|^2 + O(\log(-\log |s|)) \\ &= \frac{m}{12} \log |s^{\nu}|^2 + O(\log(-\log |s|)) \quad (s \rightarrow 0). \end{aligned}$$

This completes the proof.  $\square$

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